(2013) Cardinality

Suppose $|A| = |B|$.

Then: $\exists$ bijection $f : A \to B$.

Create fcn $g : \mathcal{P}(A) \to \mathcal{P}(B)$.

s.t. $\forall X \subseteq A$ (could have said: $\forall X \in \mathcal{P}(A)$)

$$g(X) = \{ f(x) : x \in X \}$$

Claim 1: g is injective.

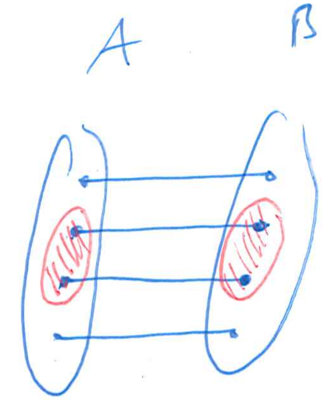
Suppose $X, Y \subseteq A$ and

$$g(X) = g(Y).$$

By def of g:

$$\{ f(x) : x \in X \} = \{ f(x) : x \in Y \}.$$

Need to show: $X = Y$.



Suppose $x \in X$. Then $f(x) \in g(X) = g(Y) = \{f(y) : y \in Y\}$.

So $\exists\, y \in Y$ st $f(x) = f(y)$.

Since f is injective, $x = y$,

so $x \in Y$.

Therefore $X \subseteq Y$.

Same method shows $Y \subseteq X$.

Therefore $X = Y$, so g is injective.

Claim 2: g is surjective.

Let $Y \subseteq B$.

Define $X = \{x \in A : f(x) \in Y\}$. Note $X \subseteq A$.

Claim: $g(X) = Y$.

If $y \in g(X)$, then $y \in Y$ by def of g.

So $g(X) \subseteq Y$.

Let $y \in Y$. Since f is surjective, $\exists x \in A$ s.t. $f(x) = y$. Then $x \in X$, so $f(x) \in g(X)$, so $y \in g(X)$, so $Y \subseteq g(X)$.

Thus g is surjective.

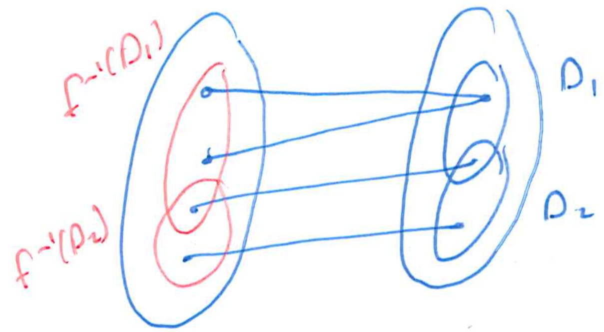
$g : \mathcal{P}(A) \to \mathcal{P}(B)$ is bijective,

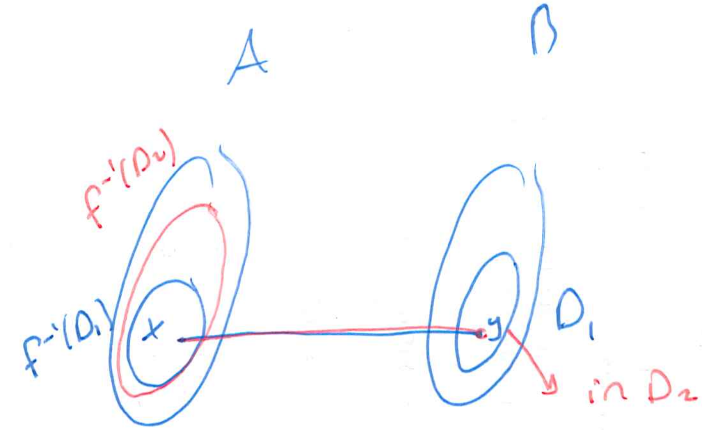
so: $|\mathcal{P}(A)| = |\mathcal{P}(B)|$. $\square$

Functions (2014)

$A \xrightarrow{f} B$

(a)





Suppose f is surjective, and

$f^{-1}(D_1) \subseteq f^{-1}(D_2)$.

Let $y \in D_1$. Since f is surjective, $\exists\, x \in A$ s.t. $f(x) = y$. Since $f(x) = y \in D_1$, then $x \in f^{-1}(D_1)$.

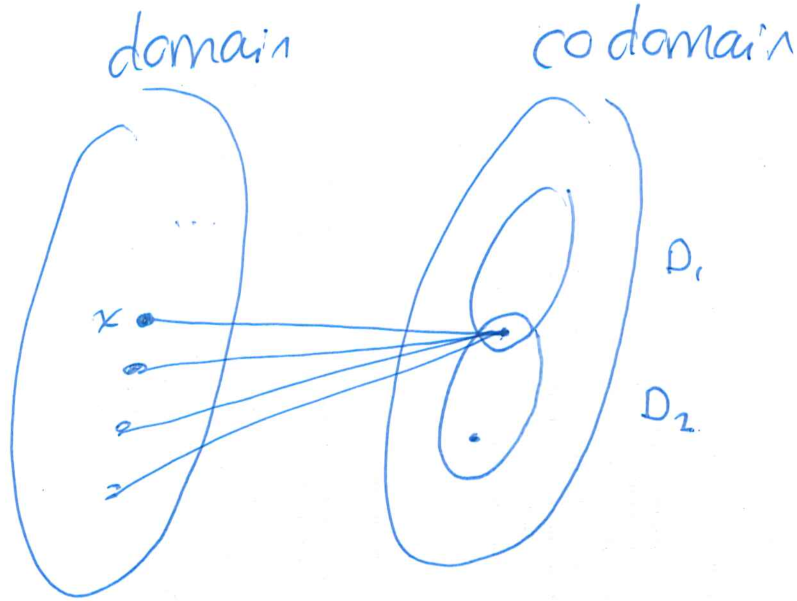
Then, since $f^{-1}(D_1) \subseteq f^{-1}(D_2)$, then $x \in f^{-1}(D_2)$.

Then: $f(x) \in D_2$. So $f(x) = y \in D_2$.

So: $D_1 \subseteq D_2$.

$\square$

(b)



$\boxed{\text{Example}}$ $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 0$ for all $x \in \mathbb{R}$.

Let $D_1 = \{0, 1\}$ and $D_2 = \{-1, 0\}$.

Note $D_1 \not\subseteq D_2$.

$f^{-1}(D_1) = \mathbb{R}$ and $f^{-1}(D_2) = \mathbb{R}$

so $f^{-1}(D_1) \subseteq f^{-1}(D_2)$.

(2013) Induction

(a) (partial idea of pf)

$\boxed{\text{Ind Step:}}$

$$\sum_{r=1}^{n+1} r(r+1) = \underbrace{\sum_{r=1}^{n} r(r+1)}_{\text{use ind. hyp.}} + \underbrace{(n+1)(n+2)}_{r=n+1}$$

$$= \frac{1}{3}n(n+1)(n+2) + (n+1)(n+2)$$

$$= (n+1)(n+2)\left(\frac{1}{3}n+1\right)$$

$$= \frac{1}{3}(n+1)(n+2)(n+3)$$

$$= \frac{1}{3}\boxed{n+1}\left(\boxed{n+1}+1\right)\left(\boxed{n+1}+2\right)$$

(b) Base Case:

If $n=1$,

$$\sum_{j=1}^{1} j^2 = 1^2 > \frac{1^3}{10} = \frac{n^3}{10}$$

Inductive Step:

Hypothesis: suppose $\sum_{j=1}^{n} j^2 > \frac{n^3}{10}$ when

$n \leq k$.

Consider $n = k+1$.

$$\sum_{j=1}^{k+1} j^2 = \sum_{j=1}^{k} j^2 + (k+1)^2 \overset{(IH)}{>} \frac{k^3}{10} + k^2+2k+1$$

$$= \frac{k^3+10k^2+20k+10}{10}$$

$$\overset{?}{>} \frac{k^3+3k^2+3k+1}{10} = \frac{(k+1)^3}{10}$$

intentional sloppiness

(2014) Induction

Ind. Step.

Hyp: If $n \le k$, and $k \ge 3$ then

$$\frac{(n+2)(n+1)n}{6} x^3 \le (1+x)^{n+2}$$

In particular, $\frac{k(k+2)(k+1)}{6} x^3 \le (1+x)^{k+2}$

Consider $n = k+1$.

$$\frac{(k+3)(k+2)(k+1)}{6} x^3 = \left(\underbrace{\frac{k(k+2)(k+1)}{6}} + \frac{3(k+2)(k+1)}{6} \right) x^3$$

$$\overset{(hyp)}{\le} (1+x)^{k+2} + \frac{3(k+2)(k+1)}{6} x^3$$

$\boxed{\text{if } k \ge 3}$

$$\le (1+x)^{k+2} + \frac{k(k+2)(k+1)}{6} x^3 \overset{(hyp)}{\le} (1+x)^{k+2} + (1+x)^{k+2}$$

⋮

$$< (1+x)^{k+2} + x(1+x)^{k+2}$$

$$(1+x)^{k+2} + x(1+x)^{k+2} = (1+x)(1+x)^{k+2} = (1+x)^{k+1+2} = (1+x)^{k+3}$$

<u>Base cases</u>

$n = 1$

$n = 2$

$n = 3$