Proving If-And-Only-If Statements

Outline:

**Proposition:** \( P \iff Q \).

**Proof:**
Proving If-And-Only-If Statements

Outline:

**Proposition:** \( P \Leftrightarrow Q \).

**Proof:**

Part 1: \( P \Rightarrow Q \).
Proving If-And-Only-If Statements

Outline:

**Proposition:** \( P \iff Q \).

**Proof:**

Part 1: \( P \implies Q \).
Part 2: \( Q \implies P \).
Outline:

**Proposition:** $P \iff Q$.

*Proof:*

Part 1: $P \implies Q$.

Part 2: $Q \implies P$.

Therefore, $P \iff Q$. 


7. Proving Nonconditional Statements

7.1 If-And-Only-If Proof

7.2 Equivalent Statements

7.3 Existence and Uniqueness Proofs

7.4 (Non-) Constructive Proofs

Outline:

**Proposition:** $P \iff Q$.

*Proof:*

Part 1: $P \implies Q$.

Part 2: $Q \implies P$.

Therefore, $P \iff Q$.

**Proposition:** $\forall a, b \in \mathbb{Z}, a \equiv b \mod 6$ if and only if $a \equiv b \mod 2$ and $a \equiv b \mod 3$. 
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Outline:

Proposition: \( P \iff Q \).

Proof:

Part 1: \( P \implies Q \).

Part 2: \( Q \implies P \).

Therefore, \( P \iff Q \).

Proposition: \( \forall a, b \in \mathbb{Z}, a \equiv b \mod 6 \) if and only if \( a \equiv b \mod 2 \) and \( a \equiv b \mod 3 \).

Proof:

Suppose \( a \equiv b \mod 6 \).
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**Proving If-And-Only-If Statements**

Outline:

**Proposition:** \( P \iff Q \).

**Proof:**

Part 1: \( P \implies Q \).
Part 2: \( Q \implies P \).

Therefore, \( P \iff Q \).  

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**Proposition:** \( \forall a, b \in \mathbb{Z}, a \equiv b \mod 6 \) if and only if \( a \equiv b \mod 2 \) and \( a \equiv b \mod 3 \).

**Proof:**

Suppose \( a \equiv b \mod 6 \). Then 6\(|(a - b)\), so \( 6x = (a - b) \) for some \( x \in \mathbb{Z} \).
Then \( (a - b) = 2(3x) = 3(2x) \), so \( (a - b) \) is divisible by both 2 and 3. Then \( a \equiv b \mod 2 \) and \( a \equiv b \mod 3 \).
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Part 1: \( P \implies Q \).

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**Proposition:** \( \forall a, b \in \mathbb{Z}, a \equiv b \mod 6 \) if and only if
\( a \equiv b \mod 2 \) and \( a \equiv b \mod 3 \).

**Proof:**

Suppose \( a \equiv b \mod 6 \). Then \( 6 \mid (a - b) \), so \( 6x = (a - b) \) for some \( x \in \mathbb{Z} \).

Then \( (a - b) = 2(3x) = 3(2x) \), so \( (a - b) \) is divisible by both 2 and 3. Then \( a \equiv b \mod 2 \) and \( a \equiv b \mod 3 \).

Suppose \( a \equiv b \mod 3 \) and \( a \equiv b \mod 2 \).
Proving If-And-Only-If Statements

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Part 1: \( P \implies Q \).

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Therefore, \( P \iff Q \). □

**Proposition:** \( \forall a, b \in \mathbb{Z}, a \equiv b \mod 6 \) if and only if \( a \equiv b \mod 2 \) and \( a \equiv b \mod 3 \).

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Suppose \( a \equiv b \mod 6 \). Then \( 6|(a - b) \), so \( 6x = (a - b) \) for some \( x \in \mathbb{Z} \). Then \( (a - b) = 2(3x) = 3(2x) \), so \( (a - b) \) is divisible by both 2 and 3. Then \( a \equiv b \mod 2 \) and \( a \equiv b \mod 3 \).

Suppose \( a \equiv b \mod 3 \) and \( a \equiv b \mod 2 \). Then \( (a - b) = 2x = 3y \) for some \( x, y \in \mathbb{Z} \). From the first expression, we see that \( (a - b) \) is even, so \( y \) is even as well. Then \( y = 2z \) for some \( z \in \mathbb{Z} \), so \( (a - b) = 3y = 6z \). Therefore \( a \equiv b \mod 6 \). □
Proposition: \( \forall a, b \in \mathbb{N}, \gcd(a, b) = b \iff b \mid a \)
Proposition: \( \forall a, b \in \mathbb{N}, \gcd(a, b) = b \iff b|a \)

(\( \Rightarrow \))
Suppose \( \gcd(a, b) = b \). Then \( b \) is a divisor of both \( a \) and \( b \). In particular, \( b|a \).

(\( \Leftarrow \))
Suppose \( b|a \). Then \( b \) is a divisor of both \( a \) and \( b \), so \( \gcd(a, b) \geq b \). Since \( b > 0 \), any \( c \in \mathbb{N} \) with \( c > b \) is not a divisor of \( b \). So, \( \gcd(a, b) \leq b \). Therefore \( \gcd(a, b) = b \).
Lists of Equivalent Statements

**Theorem:** Suppose $A$ is an $n \times n$ matrix. The following statements are equivalent:

(a) The matrix $A$ is invertible.

(b) The equation $Ax = b$ has a unique solution for every $b \in \mathbb{R}^n$.

(c) The equation $Ax = 0$ has only the trivial solution.

(d) The reduced row echelon form of $A$ is $I_n$.

(e) $\det(A) \neq 0$.

(f) The matrix $A$ does not have 0 as an eigenvalue.
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(f) The matrix $A$ does not have 0 as an eigenvalue.

\[ (a) \quad \Rightarrow \quad (b) \quad \Rightarrow \quad (c) \quad \Leftarrow \quad (d) \quad \Leftarrow \quad (e) \quad \Leftarrow \quad (f) \]
**Proposition:** For any $x, y \in \mathbb{R} - \{0\}$, the following statements are equivalent:

(A) $|x + y| = |x| + |y|

(B) $xy = |xy|

(C) $(x + y)^2 = x^2 + y^2 + 2|xy|

(D) $x$ and $y$ have the same sign
Equivalent Statements

**Proposition:** For any $x, y \in \mathbb{R} - \{0\}$, the following statements are equivalent:

- (A) $|x + y| = |x| + |y|$
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- (C) $(x + y)^2 = x^2 + y^2 + 2|xy|$
- (D) $x$ and $y$ have the same sign
Existence versus Existence and Uniqueness

**Proposition:** There exists a natural number $n$ such that $n^2 = n$.

**Proposition:** There exists a *unique* natural number $n$ such that $n^2 = n$. 
Proposition: There exists a natural number \( n \) such that \( n^2 = n \).

There is at least one.

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Existence versus Existence and Uniqueness

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There is exactly one.
Existence versus Existence and Uniqueness

**Proposition:** There exists a natural number \( n \) such that \( n^2 = n \).

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\[ \exists n \in \mathbb{N}, n^2 = n \]

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**Proposition:** There exists a natural number $n$ such that $n^2 = n$.

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$\exists n \in \mathbb{N}, n^2 = n$

**Proof:** Note that $1 \in \mathbb{N}$, and $1^2 = 1$.

**Proposition:** There exists a *unique* natural number $n$ such that $n^2 = n$.

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**Proposition:** There exists a *unique* natural number $n$ such that $n^2 = n$.

There is exactly one.

$\exists! n \in \mathbb{N}, n^2 = n$

**Proof:** Note that $1 \in \mathbb{N}$, and $1^2 = 1$. So, there exists at least one $n \in \mathbb{N}$ such that $n^2 = 1$.

Now, suppose $n \in \mathbb{N}$ and $n^2 = n$. Then $n^2 - n = 0$, so $n(n - 1) = 0$. Then $n = 1$ or $n = 0$. Since $n \in \mathbb{N}$, we conclude $n = 1$. That is, $n = 1$ is the *only* natural number such that $n^2 = n$. 
Good or Bad?

**Proposition:** Let \( f(x) = \frac{xe^x - 5x^2e^x}{x^4} \).
There exists a unique integer \( x \) such that \( f(x) = 0 \).

**Proof:** Suppose \( x \in \mathbb{Z} \) and \( f(x) = 0 \). Then:

\[
\frac{xe^x - 5x^2e^x}{x^4} = 0
\]

\[
x^4e^x - 5x^2e^x = 0
\]

\[
xe^x(1 - 5x) = 0
\]

\[
x = 0 \text{ or } e^x = 0 \text{ or } (1 - 5x) = 0
\]

\[
x = 0 \text{ or } x = \frac{1}{5}
\]

Since \( x \in \mathbb{Z} \), we conclude \( x = 0 \). That is, the only integer solution to \( f(x) = 0 \) is \( x = 0 \).
Proposition: Let \( f(x) = \frac{xe^x - 5x^2e^x}{x^4} \).
There exists a unique integer \( x \) such that \( f(x) = 0 \).

Proof: Suppose \( x \in \mathbb{Z} \) and \( f(x) = 0 \). Then:

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xe^x - 5x^2e^x = 0
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\[
x = 0 \quad \text{or} \quad e^x = 0 \quad \text{or} \quad (1 - 5x) = 0
\]

\[
x = 0 \quad \text{or} \quad x = \frac{1}{5}
\]

Since \( x \in \mathbb{Z} \), we conclude \( x = 0 \). That is, the only integer solution to \( f(x) = 0 \) is \( x = 0 \).

But \( x = 0 \) is not in the domain of \( f(x) \). IF any integer solution exists, THEN there's only one... but actually, none exists!
Proposition: There exists precisely one even prime number.
**Proposition:** There exists precisely one even prime number.

*Proof:* First, we note that 2 is an even prime number, so there exists at least one even prime number.

To show that 2 is unique, suppose $n$ is an even prime number. Since $n$ is even, $n = 2a$ for some integer $a$. Since $n$ is prime, $n$ is positive, and the only positive divisors of $n$ are 1 and $n$. But, we just showed that 2 is a positive divisor of $n$. So, $2 \in \{x \in \mathbb{N} : x|n\} = \{1, n\}$. Since $2 \neq 1$, we conclude $2 = n$. We conclude that 2 is the only even prime number.
Greatest Common Divisor

**Proposition:** Given any two coprime integers $a$ and $b$, there exist $k, \ell \in \mathbb{Z}$ such that

$$ak + b\ell = 1$$
Proposition: Given any two coprime integers $a$ and $b$, there exist $k, \ell \in \mathbb{Z}$ such that

$$ak + b\ell = 1$$

Proof:
Let $A = \{ax + by : x, y \in \mathbb{Z}\}$.
Let $d$ be the smallest positive element of $A$, and say $d = ak + b\ell$.

Claim: $d|a$.

Proof of Claim:
There exists an integer $q$ such that $a = dq + r$ for some $r \in \mathbb{Z}$, $0 \leq r < d$. (Division.)

$$r = a - dq$$
$$= a - [ak + b\ell]q$$
$$= a(1 - kq) + b(-\ell q)$$

So, $r \in A$. Since $0 \leq r < d$ (since it’s a remainder) and $r \leq 0$ or $r \geq d$, we conclude $r = 0$. That is, $d|a$.

By the same logic, $d|b$, so $d = 1$. 
Proposition: Given any two coprime integers $a$ and $b$, there exist $k, \ell \in \mathbb{Z}$ such that
\[ ak + b\ell = 1 \]

Corollary: Given any two integers $a$ and $b$, there exist $k, \ell \in \mathbb{Z}$ such that
\[ ak + b\ell = \gcd(a, b) \]
Proposition: Given any two coprime integers $a$ and $b$, there exist $k, \ell \in \mathbb{Z}$ such that

$$ak + bl = 1$$

Corollary: Given any two integers $a$ and $b$, there exist $k, \ell \in \mathbb{Z}$ such that

$$ak + bl = \gcd(a, b)$$

Proof:
Let $d = \gcd(a, b)$.

- **Claim 1:** $\frac{a}{d}, \frac{b}{d} \in \mathbb{Z}$.

- **Claim 2:** $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

By the Proposition above, there exist $k, \ell \in \mathbb{Z}$ such that

$$\left(\frac{a}{d}\right)k + \left(\frac{b}{d}\right)\ell = 1.$$  

Then, multiplying both sides of the equation by $d$,

$$ak + bl = d.$$
Proposition: Given any two coprime integers $a$ and $b$, there exist $k, \ell \in \mathbb{Z}$ such that

$$ak + b\ell = 1$$

- $a = 3, \ b = 7$: 
  
  $7(2) - 3(1) = 1$
  
  $a = 5, \ b = 8$: 
  
  $5(5) - 8(3) = 1$; also $8(2) - 5(3) = 1$
  
  $a = 22, \ b = 37$: 
  
  $37(3) - 22(5) = 111 - 110 = 1$

The proof we gave was non-constructive. It didn’t tell us what $k$ and $\ell$ were–only that they exist.
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7. Proving Nonconditional Statements

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Powers of Irrationals

**Proposition:** \( \exists x, y \in \mathbb{R} - \mathbb{Q}, \ x^y \in \mathbb{Q} \)
Powers of Irrationals

**Proposition:** \( \exists x, y \in \mathbb{R} - \mathbb{Q}, x^y \in \mathbb{Q} \)

**Proof:** (non-constructive)

Let \( r = \sqrt{2}^{\sqrt{2}} \).
Powers of Irrationals

**Proposition:** $\exists x, y \in \mathbb{R} - \mathbb{Q}, x^y \in \mathbb{Q}$

*Proof:* (non-constructive)

Let $r = \sqrt{2^{\sqrt{2}}}$.

If $r$ is rational,

If $r$ is irrational,
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Let \( r = \sqrt{2}^{\sqrt{2}} \).
If \( r \) is rational, then take \( x = y = \sqrt{2} \).
If \( r \) is irrational,

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If \( r \) is rational, then take \( x = y = \sqrt{2} \).
If \( r \) is irrational, then take \( x = r \) and \( y = \sqrt{2} \).
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\[
x^y = \left( \sqrt{2}^{\sqrt{2}} \right)^{\sqrt{2}} = \sqrt{2}^{\left(\sqrt{2}\right)\left(\sqrt{2}\right)} = \sqrt{2}^2 = 2
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**Proof:** (Constructive)

Let \( x = \sqrt{2} \) and \( y = \log_2 9 \).

Claim: \( y \notin \mathbb{Q} \).

Proof of claim: Suppose \( p/q = \log_2 9 \) for some \( p, q \in \mathbb{Z} \). Since \( \log_2 9 > 0 \), actually we can choose \( p, q \in \mathbb{N} \). Then \( 2^{p/q} = 9 = 3^2 \), so \( 2^p = 3^2 q \). Since \( p \in \mathbb{N} \), the left-hand side of the equation is even. But the right-hand side is the product of odd numbers, so it is odd. Then an even number is equal to an odd number, which is a contradiction.

We conclude \( y \notin \mathbb{Q} \).

\( x^y = \sqrt{2} \log_2 9 = \sqrt{2} \log_2 3 = 3 \).
Powers of Irrationals

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**Proof:** (Constructive)

Let $x = \sqrt{2}$ and $y = \log_2 9$.

**Claim:** $y \notin \mathbb{Q}$.

**Proof of claim:** Suppose $\frac{p}{q} = \log_2 9$ for some $p, q \in \mathbb{Z}$. Since $\log_2 9 > 0$, actually we can choose $p, q \in \mathbb{N}$.

Then $2^{p/q} = 9 = 3^2$, so $2^p = 3^{2q}$. Since $p \in \mathbb{N}$, the left-hand side of the equation is even. But the right-hand side is the product of odd numbers, so it is odd. Then an even number is equal to an odd number, which is a contradiction.

**We conclude** $y \notin \mathbb{Q}$. 
Powers of Irrationals

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\[
x^y = \sqrt{2}^{\log_2 9} = \sqrt{2}^{2 \log_2 3} = 2^{\log_2 3} = 3
\]