5. Contra-positive Proof

5.1 Contra-positive Proof

5.2 Congruence of Integers

5.3 Mathematical Writing

How to prove \( P \Rightarrow Q \):

Recall this statement is equivalent to \( \sim Q \Rightarrow \sim P \).
How to prove $P \Rightarrow Q$:
Recall this statement is equivalent to $\sim Q \Rightarrow \sim P$

**Proposition**: If $P$, then $Q$.

**Proof**: Suppose $\sim Q$.

$\therefore$
Therefore $\sim P$. 

\qed
**Proposition:** Let \( x, y \in \mathbb{Z} \). If \( x + y \) is even, then \( x \) and \( y \) have the same parity.
**Proposition:** Let $x, y \in \mathbb{Z}$. If $x + y$ is even, then $x$ and $y$ have the same parity.

**Proposition:** Let $x, y \in \mathbb{Z}$. If $x + y$ is odd, then $x \neq y$. 
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Proposition: For any \( n \in \mathbb{Z} \), \( n^2 \) is even if and only if \( n \) is even.
Proposition: Let $x, y \in \mathbb{Z}$. If $x + y$ is even, then $x$ and $y$ have the same parity.

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Proposition: For any $n \in \mathbb{Z}$, $n^2$ is even if and only if $n$ is even.

Proposition: Let $d, x, y \in \mathbb{Z}$. If $d \nmid (xy)$, then $d \nmid x$ and $d \nmid y$. 
Proposition: Let \(x, y \in \mathbb{Z}\). If \(x + y\) is even, then \(x\) and \(y\) have the same parity.

Suppose \(x\) and \(y\) have opposite parity. WLOG, let \(x\) be even, so \(y\) is odd. Then there exist integers \(a\) and \(b\) such that \(x = 2a\) and \(y = 2b + 1\). Then \(x + y = 2a + 2b + 1 = 2(a + b) + 1\). Since \(a + b\) is an integer, \(x + y\) is odd.

Proposition: Let \(x, y \in \mathbb{Z}\). If \(x + y\) is odd, then \(x \neq y\).

Suppose \(x = y\). Then \(x + y = 2x\). Since \(x \in \mathbb{Z}\), that means \(x + y\) is even.

Proposition: For any \(n \in \mathbb{Z}\), \(n^2\) is even if and only if \(n\) is even.

Suppose \(n\) is even. Then \(n = 2a\) for some \(a \in \mathbb{Z}\), so \(n^2 = 4a^2 = 2(2a^2)\). Since \(2a^2 \in \mathbb{Z}\), \(a^2\) is even.

Now, suppose \(n\) is odd. Then \(n = 2a + 1\) for some \(a \in \mathbb{Z}\), so \(n^2 = 4a^2 + 4a + 1 = 2(2a^2 + 2a) + 1\). Since \(2a^2 + 2a \in \mathbb{Z}\), \(n^2\) is odd.

Proposition: Let \(d, x, y \in \mathbb{Z}\). If \(d \nmid (xy)\), then \(d \nmid x\) and \(d \nmid y\).

Suppose \(d \mid x\) or \(d \mid y\): WLOG, \(d \mid x\). Then \(da = x\) for some integer \(a\), so \(xy = day = d(ay)\). Therefore, \(d\) divides \(xy\).
### Congruence

#### Definition

Given $a, b \in \mathbb{Z}$ and $n \in \mathbb{N}$, we say that $a$ and $b$ are **congruent modulo $n$** if $n|(a - b)$. We express this as

$$a \equiv b \pmod{n}.$$ 

If $a$ and $b$ are not congruent modulo $n$, we write this as $a \not\equiv b \pmod{n}$.
Congruence

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- $7 \equiv 17 \mod 5$
## Congruence

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- \( 7 \equiv 17 \mod 5 \)
- \( -5 \equiv 10 \mod 5 \)
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Definition

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If $a$ and $b$ are not congruent modulo $n$, we write this as $a \not\equiv b \mod n$.

- $7 \equiv 17 \mod 5$
- $-5 \equiv 10 \mod 5$
- $1 \not\equiv 3 \mod 4$
- $\forall a, b \in \mathbb{Z}, \quad a \equiv a \mod b$
Examples

For any $a, b \in \mathbb{Z}$ and any $n \in \mathbb{N}$, $a \equiv b \mod n \implies a^2 \equiv b^2 \mod n$.

For any $n, c \in \mathbb{N}$, $35c \not\equiv 72c \mod n \implies n \nmid c$. 
- Since $x = y$, it's negative.
Since $x = y$, it's negative. What is "it", $x$ or $y$? Better to specify—avoid ambiguities.
Since $x = y$, it’s negative.
What is ”it”, $x$ or $y$? Better to specify–avoid ambiguities.

Since $x = y$, $x$ is odd.
Since $x = y$, it's negative.
What is ”it”, $x$ or $y$? Better to specify–avoid ambiguities.
Since $x = y$, $x$ is odd.

Every prime number $x$ that is greater than $2$ is odd.
5. Contra-positive Proof

5.1 Con-trapositive Proof

5.2 Con-gruence of Integers

5.3 Mathe-matical Writing

Make These Better

- Since \( x = y \), it's negative.
  What is "it", \( x \) or \( y \)? Better to specify–avoid ambiguities.
  Since \( x = y \), \( x \) is odd.

- Every prime number \( x \) that is greater than 2 is odd.
  It’s not necessary to call that number \( x \), since we never use its name again. Don’t introduce any symbols you won’t use.
5. Contra-positive Proof

5.1 Con-tra-apositive Proof

5.2 Con-gruence of Integers

5.3 Mathematical Writing

Make These Better

- Since $x = y$, it’s negative. What is ”it”, $x$ or $y$? Better to specify–avoid ambiguities. Since $x = y$, $x$ is odd.

- Every prime number $x$ that is greater than 2 is odd. It’s not necessary to call that number $x$, since we never use its name again. Don’t introduce any symbols you won’t use. Every prime number that is greater than 2 is odd.
Make These Better

■ Since \( x = y \), it’s negative.
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■ If the two numbers are \( = \), then they are both odd.
Since $x = y$, it’s negative.
What is ”it”, $x$ or $y$? Better to specify–avoid ambiguities.

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If the two numbers are $=$, then they are both odd.
It’s awkward to read symbols as words outside the context of an equation.
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- If the two numbers are $=\,$, then they are both odd.
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  If the two numbers are equal, then they are both odd.
Make These Better

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- If the two numbers are =, then they are both odd.
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  If the two numbers are equal, then they are both odd.

- Since \( a^2 | a \), \( a \neq 0 \), \( a = 0 \) or \( a = 1 \).
Make These Better

- Since \( x = y \), it's negative.
  What is "it", \( x \) or \( y \)? Better to specify–avoid ambiguities.
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- Since \( a^2 | a \), \( a \neq 0 \), \( a = 0 \) or \( a = 1 \).
  Avoid ambiguity by separating clauses with mathematical expressions with words.
Make These Better

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- If the two numbers are \( = \), then they are both odd.
  It’s awkward to read symbols as words outside the context of an equation.
  If the two numbers are equal, then they are both odd.

- Since \( a^2 | a, a \neq 0 \), \( a = 0 \) or \( a = 1 \).
  Avoid ambiguity by separating clauses with mathematical expressions with words.
  Since \( a^2 | a \) and \( a \neq 0 \), we conclude that \( a = 0 \) or \( a = 1 \).

- Since \( a \) is odd, \( a = 2x + 1 \).
Make These Better

- Since $x = y$, it's negative.
  What is ”it”, $x$ or $y$? Better to specify–avoid ambiguities.
  Since $x = y$, $x$ is odd.

- Every prime number $x$ that is greater than 2 is odd.
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- If the two numbers are $=$, then they are both odd.
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- Since $a^2 | a$, $a \neq 0$, $a = 0$ or $a = 1$.
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  Since $a^2 | a$ and $a \neq 0$, we conclude that $a = 0$ or $a = 1$.

- Since $a$ is odd, $a = 2x + 1$.
  Explain each new symbol.
Make These Better

- **Since** $x = y$, it’s negative.
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- If the two numbers are $=$, then they are both odd.
  It’s awkward to read symbols as words outside the context of an equation.
  If the two numbers are equal, then they are both odd.

- **Since** $a^2 | a$, $a \nless 0$, $a = 0$ or $a = 1$.
  Avoid ambiguity by separating clauses with mathematical expressions with words.
  Since $a^2 | a$ and $a \nless 0$, we conclude that $a = 0$ or $a = 1$.

- **Since** $a$ is odd, $a = 2x + 1$.
  Explain each new symbol.
  Since $a$ is odd, $a = 2x + 1$ for some $x \in \mathbb{Z}$.

Read Chapter 5.3 for more!
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