Functions as Relations

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\( xRy \) if and only if \( x^2 = y \)
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**Function**

Suppose \(A\) and \(B\) are sets. A **function** \(f\) from \(A\) to \(B\) (denoted as \(f : A \rightarrow B\)) is a relation \(f \subseteq A \times B\), satisfying the property that for each \(a \in A\) the relation \(f\) contains exactly one ordered pair of the form \((a, b)\). (Functions pass the vertical line test.)

The statement \((a, b) \in f\) is abbreviated \(f(a) = b\).
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*(Functions pass the vertical line test.)*

The statement \( (a, b) \in f \) is abbreviated \( f(a) = b \).

\( \{(1, 2), (2, 4), (3, 6), (2.7, 5.4), (\pi, 2\pi), \ldots\} \)

\( \{(1, -1), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5), \ldots\} \)
\[ f(x) = x^2 \]

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\( f : \mathbb{R} \to \mathbb{R}, \; f(x) = 2x \)

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Not a function: \( f(1) \) has two values.
Function Vocabulary

**Function**

Let $f : A \rightarrow B$ be a function. We call $A$ the \textbf{domain} of $f$, and $B$ the \textbf{codomain} of $f$. The \textbf{range} of $f$ is $\{f(a) : a \in A\}$. 
Function Vocabulary

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Let $f : A \rightarrow B$ be a function. We call $A$ the **domain** of $f$, and $B$ the **codomain** of $f$. The **range** of $f$ is $\{ f(a) : a \in A \}$.

- $f : (0, \infty) \rightarrow \mathbb{R}$, \quad $f(x) = \ln x$
- $f : [1, \infty) \rightarrow \mathbb{R}$, \quad $f(x) = \sqrt{1 - x}$
- $f : \mathbb{Z} \rightarrow \mathbb{Z}$, \quad $f(x) = |x|$
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$$f : (0, \infty) \rightarrow \mathbb{R}, \quad f(x) = \ln x$$
Codomain: $\mathbb{R}$, Range: $\mathbb{R}$

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---

**Function Vocabulary**

\[
\begin{align*}
    f &: (0, \infty) \rightarrow \mathbb{R}, & f(x) &= \ln x \\
    f &: [1, \infty) \rightarrow \mathbb{R}, & f(x) &= \sqrt{1 - x} \\
    f &: \mathbb{Z} \rightarrow \mathbb{Z}, & f(x) &= |x|
\end{align*}
\]

Codomain: $\mathbb{R}$, Range: $\mathbb{R}$

Codomain: $\mathbb{R}$, Range: $[0, \infty)$
**Function Vocabulary**

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$$f : (0, \infty) \to \mathbb{R}, \quad f(x) = \ln x \quad \text{Codomain: } \mathbb{R}, \text{Range: } \mathbb{R}$$

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$$f : \mathbb{Z} \to \mathbb{Z}, \quad f(x) = |x| \quad \text{Codomain: } \mathbb{Z}, \text{Range: } \mathbb{N} \cup \{0\}$$
Function Vocabulary

Function

Let \( f : A \rightarrow B \) be a function. We call \( A \) the \textbf{domain} of \( f \), and \( B \) the \textbf{codomain} of \( f \). The \textbf{range} of \( f \) is \( \{ f(a) : a \in A \} \).

- \( f : (0, \infty) \rightarrow \mathbb{R}, \quad f(x) = \ln x \)  
  Codomain: \( \mathbb{R} \), Range: \( \mathbb{R} \)

- \( f : [1, \infty) \rightarrow \mathbb{R}, \quad f(x) = \sqrt{1 - x} \)  
  Codomain: \( \mathbb{R} \), Range: \( [0, \infty) \)

- \( f : \mathbb{Z} \rightarrow \mathbb{Z}, \quad f(x) = |x| \)  
  Codomain: \( \mathbb{Z} \), Range: \( \mathbb{N} \cup \{0\} \)

The range is a subset of the codomain.
Functions that aren’t from the reals to the reals
Functions that aren’t from the reals to the reals

\[ p : (\text{location}) \rightarrow (\text{color}) \]
Functions that aren’t from the reals to the reals

\[ p : \text{(location)} \rightarrow \text{(color)} \]

<p>| | | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>(1, 6)</td>
<td>(2, 6)</td>
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<td>(2, 2)</td>
<td>(3, 2)</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>(2, 1)</td>
<td>(3, 1)</td>
</tr>
</tbody>
</table>
Functions that aren’t from the reals to the reals

\[ p : (\text{location}) \rightarrow (\text{color}) \]

\[
\begin{array}{ccc}
(1, 6) & (2, 6) & (3, 6) \\
(1, 5) & (2, 5) & (3, 5) \\
(1, 4) & (2, 4) & (3, 4) \\
(1, 3) & (2, 3) & (3, 3) \\
(1, 2) & (2, 2) & (3, 2) \\
(1, 1) & (2, 1) & (3, 1)
\end{array}
\]

\[ p : (\{1, 2, 3\} \times \{1, 2, 3, 4, 5, 6\}) \rightarrow \{ \} \]
Functions that aren’t from the reals to the reals

\[ p : \text{(location)} \rightarrow \text{(color)} \]

\[ p : (\{1, 2, 3\} \times \{1, 2, 3, 4, 5, 6\}) \rightarrow \{\text{white, mauve, tan, green}\} \]
Functions that aren’t from the reals to the reals

\[ p : (\text{location}) \rightarrow (\text{color}) \]

\[
\begin{array}{ccc}
(1, 6) & (2, 6) & (3, 6) \\
(1, 5) & (2, 5) & (3, 5) \\
(1, 4) & (2, 4) & (3, 4) \\
(1, 3) & (2, 3) & (3, 3) \\
(1, 2) & (2, 2) & (3, 2) \\
(1, 1) & (2, 1) & (3, 1) \\
\end{array}
\]

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\[ p = \{ ((1, 1), \text{white}) , ((1, 2), \text{white}) , ((2, 1), \text{green}) , \ldots \} \]
Functions that aren’t from the reals to the reals

\[ p : (\text{location}) \rightarrow (\text{color}) \]

\[
\begin{array}{ccc}
(1, 6) & (2, 6) & (3, 6) \\
(1, 5) & (2, 5) & (3, 5) \\
(1, 4) & (2, 4) & (3, 4) \\
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### Functions that aren’t from the reals to the reals

A function $p$ maps locations to colors. Here is an example:

$$p : \text{(location)} \rightarrow \text{(color)}$$

<table>
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<tr>
<th>Location</th>
<th>Color</th>
</tr>
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<tbody>
<tr>
<td>(1, 6)</td>
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<td>(1, 4)</td>
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</tr>
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<td>(3, 3)</td>
</tr>
<tr>
<td>(1, 2)</td>
<td>(3, 2)</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>(3, 1)</td>
</tr>
</tbody>
</table>

$p : (\{1, 2, 3\} \times \{1, 2, 3, 4, 5, 6\}) \rightarrow \{\text{white, mauve, tan, green}\}$

$p = \{((1, 1), \text{white}), ((1, 2), \text{white}), ((2, 1), \text{green}), \ldots\}$
Functions that aren’t from the reals to the reals

\[ p : (\text{location}) \rightarrow (\text{color}) \]

\[
\begin{array}{ccc}
(1, 6) & (2, 6) & (3, 6) \\
(1, 5) & (2, 5) & (3, 5) \\
(1, 4) & (2, 4) & (3, 4) \\
(1, 3) & (2, 3) & (3, 3) \\
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Equality

Two functions $f : A \to B$ and $g : X \to Y$ are equal if (and only if):

- $A = X$, and
- $f(a) = g(a)$ for every $a \in A$. 

Equality in Functions
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Equal or Not?

- $f : \mathbb{N} \rightarrow \mathbb{N}: f(x) = x^2$
- $g : \mathbb{R} \rightarrow \mathbb{R}: g(x) = x^2$
- $f : \mathbb{Z} \rightarrow \mathbb{Z}: f(x) = x^2$
- $g : \mathbb{Z} \rightarrow \mathbb{R}: g(x) = |x| \ast |x|$
- $f : \{1, 2\} \rightarrow \{1, 2\}: f = \{(1, 1), (2, 1)\}$
- $g : \{1, 2\} \rightarrow \{1, 2\}: g = \{(2, 1), (1, 1)\} $
Equality

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- \( f : \mathbb{N} \to \mathbb{N} : f(x) = x^2 \) \( f \neq g \):
- \( g : \mathbb{R} \to \mathbb{R} : g(x) = x^2 \) \( g(-1) = 1 \), while \( f \) is not defined at \( x = -1 \).
- \( f : \mathbb{Z} \to \mathbb{Z} : f(x) = x^2 \)
- \( g : \mathbb{Z} \to \mathbb{R} : g(x) = |x| \cdot |x| \)

- \( f : \{1, 2\} \to \{1, 2\} : f = \{(1, 1), (2, 1)\} \)
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Two functions \( f : A \to B \) and \( g : X \to Y \) are \textcolor{orange}{\textbf{equal}} if (and only if):

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- \( f(a) = g(a) \) for every \( a \in A \).

Equal or Not?

\begin{itemize}
  \item \( f : \mathbb{N} \to \mathbb{N} : f(x) = x^2 \quad f \neq g: \)
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  \item \( f : \mathbb{Z} \to \mathbb{Z} : f(x) = x^2 \)
  \item \( g : \mathbb{Z} \to \mathbb{R} : g(x) = |x| \times |x| \quad f = g \)
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- $f : \mathbb{Z} \to \mathbb{Z}: f(x) = x^2$

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### Definition

Let $f$ be a function from $A$ to $B$.

- $f$ is **injective** (also called *one-to-one*) if, for every $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$. 

- $f$ is **surjective** (also called *onto*) if, for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. (That is: the codomain equals the range.)

- $f$ is **bijective** if it is both injective and surjective.
Injective, Surjective, Bijective

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**Definition**

Let $f$ be a function from $A$ to $B$.

- **$f$ is injective** (also called *one-to-one*) if, for every $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$.

- **$f$ is surjective** (also called *onto*) if, for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. (That is: the codomain equals the range.)

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![Diagram showing injective, surjective, and non-injective, non-surjective functions]
Injective, Surjective, Bijective

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![Diagram](image-url)
Let $f$ be a function from $A$ to $B$.

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\[ f : \mathbb{R} \rightarrow \mathbb{R} \]

\[ f(x) = x^2 \]

\[ f : \mathbb{R} \rightarrow \mathbb{R} \]

\[ f(x) = x^3 \]

neither
Injective, Surjective, Bijective

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\[
\begin{align*}
\text{Let } f &: \mathbb{R} \to [0, \infty) \\
&\quad \text{satisfy } f(x) = x^2 \\
&\quad \text{surjective}
\end{align*}
\]

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\text{Let } f &: \mathbb{R} \to \mathbb{R} \\
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Let \( f \) be a function from \( A \) to \( B \).

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\[
\begin{align*}
\text{Injective:} & & \text{Surjective:} \\
& f(x) &= x \sin x & f(x) &= \arctan x
\end{align*}
\]
Injective, Surjective, Bijective

Let $f$ be a function from $A$ to $B$.

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\[
\begin{align*}
\text{surjective} \\
\text{not injective}
\end{align*}
\]
Injective, Surjective, Bijective

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\[ f : \mathbb{R} \to \mathbb{R} \]

\[ f(x) = x \sin x \]

\[ f : \mathbb{R} \to \mathbb{R} \]

\[ f(x) = \arctan x \]
Injective, Surjective, Bijective

Let $f$ be a function from $A$ to $B$.

- $f$ is **injective** (also called *one-to-one*) if, for every $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$.

- $f$ is **surjective** (also called *onto*) if, for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. (That is: the codomain equals the range.)

- $f$ is **bijective** if it is both injective and surjective.

\[ f : \mathbb{R} \rightarrow \mathbb{R} \]
\[ f(x) = x \sin x \]

\[ f : \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2}) \]
\[ f(x) = \arctan x \]

**surjective**

**not injective**

**injective**

**surjective**
Prove or disprove:
The function \( f(x) = 3x^4 + 1, \, f : \mathbb{R} \to \mathbb{R}, \) is injective.

Prove or disprove:
The function \( f(x) = 3x^4 + 1, \, f : \mathbb{R} \to \mathbb{R}, \) is surjective.
Prove or disprove:
The function \( f(x) = 3x^4 + 1, \ f : \mathbb{R} \to \mathbb{R} \), is injective.

We notice 1 and \(-1\) are in \(\mathbb{R}\), \(1 \neq -1\), and \(f(1) = f(-1) = 4\).
This shows that \(f\) is not injective.

Prove or disprove:
The function \( f(x) = 3x^4 + 1, \ f : \mathbb{R} \to \mathbb{R} \), is surjective.
Prove or disprove: The function \( f(x) = 3x^4 + 1, \ f : \mathbb{R} \to \mathbb{R} \), is injective.

We notice 1 and \(-1\) are in \(\mathbb{R}\), \(1 \neq -1\), and \(f(1) = f(-1) = 4\). This shows that \(f\) is not injective.

Prove or disprove: The function \( f(x) = 3x^4 + 1, \ f : \mathbb{R} \to \mathbb{R} \), is surjective.

We notice 0 is in \(\mathbb{R}\), but no real value of \(x\) gives \(f(x) = 0\). This shows that \(f\) is not surjective.
Prove that the function

\[ f(x) = \frac{x + 1}{x - 1}, \quad f : (\mathbb{R} - \{1\}) \to (\mathbb{R} - \{1\}) \]

is bijective.
Prove that the function

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Proving Injectivity and Surjectivity

Prove that the function
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is bijective.

**Injectivity** Suppose \( f(x) = f(y) \) for some \( x, y \in \mathbb{R} - \{1\} \).

So, \( x = y \). We conclude \( f(x) \) is injective.

**Surjectivity**
Proving Injectivity and Surjectivity

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**Injectivity**

Suppose \( f(x) = f(y) \) for some \( x, y \in \mathbb{R} - \{1\} \).

That is:

\[ \frac{x + 1}{x - 1} = \frac{y + 1}{y - 1} \]

So:

\[ (x + 1)(y - 1) = (x - 1)(y + 1) \]

So:

\[ xy - x + y - 1 = xy + x - y - 1 \]

So:

\[ -x + y = x - y \]

So:

\[ 2y = 2x \]

So, \( x = y \). We conclude \( f(x) \) is injective.

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Proving Injectivity and Surjectivity

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**Injectivity**

**Surjectivity** Let \( y \in \mathbb{R} - \{1\} \), and define \( a = \)

Then \( a \in \mathbb{R} - \{1\} \), and \( f(a) = y \). We conclude \( f(x) \) is surjective.
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**Injectivity**

Let \( y \in \mathbb{R} - \{1\} \), and define \( a = \frac{y + 1}{y - 1} \).

We note that, since \( y \neq 1 \), \( a \in \mathbb{R} \). Furthermore, since the equation \( \frac{y + 1}{y - 1} = 1 \)

is equivalent to the equation \( y + 1 = y - 1 \), and so \( 1 = -1 \), which has no solutions, we see that \( a \neq 1 \). We note further:

\[
\begin{align*}
f(a) &= \frac{a + 1}{a - 1} = \frac{\frac{y + 1}{y - 1} + 1}{\frac{y + 1}{y - 1} - 1} = \frac{\frac{y + 1 + y - 1}{y - 1}}{\frac{y + 1 - (y - 1)}{y - 1}} = \frac{y + 1 + y - 1}{y + 1 - (y - 1)} = \frac{2y}{2} = y
\end{align*}
\]

Then \( a \in \mathbb{R} - \{1\} \), and \( f(a) = y \). We conclude \( f(x) \) is surjective.
Consider the function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by

\[
f(x, y) = (x + y, x - y)
\]

Is \( f \) injective? Is \( f \) surjective? Prove your answers.
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$f$ is injective. Suppose there exist $(x, y) \in \mathbb{R}^2$ and $(a, b) \in \mathbb{R}^2$ such that $f(x, y) = f(a, b)$. Then: (1) $x + y = a + b$, and (2) $x - y = a - b$. Adding (1) and (2), we see $2x = 2a$, so $x = a$. Then from (1) we see $y = b$. So, $(x, y) = (a, b)$.

$f$ is also surjective.
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\( f \) is injective. Suppose there exist \((x, y) \in \mathbb{R}^2\) and \((a, b) \in \mathbb{R}^2\) such that \(f(x, y) = f(a, b)\). Then: (1) \(x + y = a + b\), and (2) \(x - y = a - b\). Adding (1) and (2), we see \(2x = 2a\), so \(x = a\). Then from (1) we see \(y = b\). So, \((x, y) = (a, b)\).

\( f \) is also surjective. Given any \((a, b) \in \mathbb{R}^2\), \((\frac{a+b}{2}, \frac{a-b}{2}) \in \mathbb{R}^2\), and \(f\left(\frac{a+b}{2}, \frac{a-b}{2}\right) = (a, b)\).
Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = (x - y, x^2 - y^2)$$

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Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

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$f$ is not injective, because $f(0, 0) = f(1, 1) = (0, 0)$.

$f$ is not surjective.
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$$f(x, y) = (x - y, x^2 - y^2)$$


$f$ is not injective, because $f(0, 0) = f(1, 1) = (0, 0)$.

$f$ is not surjective. We note $(0, 1) \in \mathbb{R}^2$. If $f(x, y) = (0, 1)$, then $x - y = 0$, so $x = y$. But then $x^2 - y^2 = 0$, so $f(x, y) = (0, 0) \neq (0, 1)$. 
Consider a function $f : A \to B$, where $|A| = 3$ and $|B| = 2$.

Is $f$ injective? Is $f$ surjective?
Consider a function $f : A \rightarrow B$, where $|A| = 3$ and $|B| = 2$.

Is $f$ injective? Is $f$ surjective?

$f$ is not injective, because there are three different inputs and only two different outputs.

$f$ may or may not be surjective.
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### Pigeonhole Principle

Suppose $p$ pigeons are sitting in $h$ pigeonholes.

- If $p > h$, then some hole has more than one pigeon in it.
- If $p < h$, then some hole does not have a pigeon in it.
Proving Injectivity and Surjectivity

Consider a function $f : A \rightarrow B$, where $|A| = 3$ and $|B| = 2$.

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- If $p < h$, then some hole does not have a pigeon in it.

Let $A$ and $B$ be finite sets, and let $f : A \rightarrow B$ be a function.

- If $|A| > |B|$, then $f$ is not
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Socks and Pigeons

Suppose you have a drawer full of red, blue, and green socks. You’re grabbing clothes in the dark. How many socks do you have to take before you’re guaranteed to have a matching pair?

You roll a handful of dice all at once. You are guaranteed to have three dice showing the same number. How many dice did you roll?
The average person has about 100,000 hair follicles on their head\(^1\)
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Suppose everyone in Canada has between 0 and 999,999 hairs on their head. The population of Canada is more than 30 million. What does the Pigeonhole Principle tell us about this situation?

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(Actually, there are at least 30 people in Canada with exactly the same number of hairs on their head.)

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(Actually, there are at least 30 people in Canada with exactly the same number of hairs on their head.)

This is a non-constructive proof.

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Pigeonhole Principle

Pick six distinct numbers from the set \( \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \).
Pigeonhole Principle

Pick six distinct numbers from the set \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\).
Among your picks, there are two numbers that sum to 11.
Pigeonhole Principle

Pick six distinct numbers from the set \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.

Among your picks, there are two numbers that sum to 11.

Make the following holes:

\[\{1, 10\}, \quad \{2, 9\}, \quad \{3, 8\}, \quad \{4, 7\}, \quad \{5, 6\}\]
Pigeonhole Principle

Pick six distinct numbers from the set \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.

Among your picks, there are two numbers that sum to 11.

Make the following holes:

\{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}

Notice every number from the original set of ten appears in one of the holes. By picking six numbers, each of which fits into one of the 5 holes, you must have chosen at least two numbers from the same hole. Their sum is 11.
Pigeonhole Principle

Pick six distinct numbers from the set \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.

Among your picks, there are two numbers that sum to 11.

Make the following holes:

\{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}

Notice every number from the original set of ten appears in one of the holes. By picking six numbers, each of which fits into one of the 5 holes, you must have chosen at least two numbers from the same hole. Their sum is 11.

Suppose you have a pack of cards with the face cards removed. That is, you have 40 cards: four of each suit (♦, ♥, ♣, ♠), with face values 1-10. What is the minimum number of cards you can draw, and still be guaranteed that there are two cards of the same suit that add to 11?
**Proposition:** For any $a \in \mathbb{N}$, there exist distinct $k, \ell \in \mathbb{N}$ such that $a^k - a^\ell$ is divisible by 7.
Proposition: For any \( a \in \mathbb{N} \), there exist distinct \( k, \ell \in \mathbb{N} \) such that \( a^k - a^\ell \) is divisible by 7.

Example: if \( a = 2 \), then \( 2^4 - 2^1 = 16 - 2 = 14 = 2 \times 7 \).
Proposition: For any \( a \in \mathbb{N} \), there exist distinct \( k, \ell \in \mathbb{N} \) such that \( a^k - a^\ell \) is divisible by 7.

Example: if \( a = 2 \), then \( 2^4 - 2^1 = 16 - 2 = 14 = 2 \times 7 \).

Example: if \( a = 3 \), then \( 3^7 - 3^1 = 2187 - 3 = 2184 = 312 \times 7 \).
Pigeonhole Principle

**Proposition:** For any \( a \in \mathbb{N} \), there exist distinct \( k, \ell \in \mathbb{N} \) such that \( a^k - a^\ell \) is divisible by 7.

Example: if \( a = 2 \), then \( 2^4 - 2^1 = 16 - 2 = 14 = 2 \times 7 \).

Example: if \( a = 3 \), then \( 3^7 - 3^1 = 2187 - 3 = 2184 = 312 \times 7 \)

Proof: There are seven congruence classes mod 7. For every \( x \in \mathbb{N} \), \( a^x \) is in one of them. Since there are more than seven values of \( a^x \) when we let \( x \in \mathbb{N} \), two of them must be in the same congruence class mod 7. Then their difference is divisible by 7.

Actually: there must exist \( k, \ell \) in \( \{1, \ldots, 8\} \) such that \( a^k - a^\ell \) is divisible by 7.
Pigeonhole Principle

**Proposition:** In any set of four distinct natural numbers, there exist two distinct natural numbers whose sum or difference is divisible by 5.
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Example: In the set $\{1, 2, 3, 5\}$, $2 + 3$ is divisible by 5.
Proposition: In any set of four distinct natural numbers, there exist two distinct natural numbers whose sum or difference is divisible by 5.

Example: In the set \{1, 2, 3, 5\}, 2 + 3 is divisible by 5.
Example: In the set \{1, 6, 8, 10\}, 6 − 1 is divisible by 5.
Pigeonhole Principle

**Proposition:** In any set of four distinct natural numbers, there exist two distinct natural numbers whose sum or difference is divisible by 5.

Example: In the set \{1, 2, 3, 5\}, \(2 + 3\) is divisible by 5.

Example: In the set \{1, 6, 8, 10\}, \(6 - 1\) is divisible by 5.

**Proof** We will use the pigeonhole principle. Let the natural numbers be partitioned into three “holes” by their equivalence class mod 5, as follows:

\[ [0], \quad [1] \cup [4], \quad [2] \cup [3] \]

By the pigeonhole principle, if we pick four natural numbers, two of them must be in the same hole. If they are in the same equivalence class mod 5, then their difference is divisible by 5. If they are in different equivalence classes (but the same hole), then their sum is divisible by 5.
Function Composition

Let $f$ and $g$ be functions, $f : B \rightarrow C$ and $g : A \rightarrow B$. Then we defined the **composition** of $f$ and $g$ to be

$$(f \circ g)(x) = f(g(x))$$
## Function Composition

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Suppose $f$ and $g$ are functions from the reals to the reals, $f(x) = x^2 + 5x$, $g(x) = \sin x$. Then:

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Suppose \( f \) and \( g \) are functions from the reals to the reals, \( f(x) = x^2 + 5x \), \( g(x) = \sin x \). Then:

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f \circ g(x) = (\sin x)^2 + 5 \sin x
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Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $g : \mathbb{R} \rightarrow \mathbb{R}^2$, with

$f(x, y) = (x^2, x + y, y^2)$, and $g(x) = (3x, 4x)$.

$f \circ g$
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$$f \circ g(x) = f(3x, 4x) = (9x^2, 7x, 16x^2)$$
Composition and Injectivity, Surjectivity

For the following characteristics, give an example of functions $f$ and $g$ that have them, or show that none exists.

Assume that the range of $g$ is the same as the domain of $f$.

- $f$ injective, $f \circ g$ not injective.

- $f$ injective, $g$ injective, $f \circ g$ not injective.

- $g$ surjective, $f \circ g$ not surjective.

- $f$ surjective, $g$ surjective, $f \circ g$ not surjective.
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- $f$ injective, $f \circ g$ not injective.
  
  $f(x) = x$, $g(x) = x^2$

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- $f$ injective, $g$ injective, $f \circ g$ not injective.
  
  Suppose $f$ and $g$ are both injective. Then if $f(g(x)) = f(g(y))$, by injectivity of $f$, $g(x) = g(y)$. Then, by injectivity of $g$, $g(x) = g(y)$ implies $x = y$. So $f \circ g$ is injective.

- $g$ surjective, $f \circ g$ not surjective.
  
  $f(x) = \sin x$, $g(x) = x$

- $f$ surjective, $g$ surjective, $f \circ g$ not surjective.
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For the following characteristics, give an example of functions \( f \) and \( g \) that have them, or show that none exists.

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- \( f \) injective, \( f \circ g \) not injective.
  \[
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- \( f \) injective, \( g \) injective, \( f \circ g \) not injective.
  Suppose \( f \) and \( g \) are both injective. Then if \( f(g(x)) = f(g(y)) \), by injectivity of \( f \), \( g(x) = g(y) \). Then, by injectivity of \( g \), \( g(x) = g(y) \) implies \( x = y \). So \( f \circ g \) is injective.

- \( g \) surjective, \( f \circ g \) not surjective.
  \[
  f(x) = \sin x, \quad g(x) = x
  \]

- \( f \) surjective, \( g \) surjective, \( f \circ g \) not surjective.
  Suppose \( f \) and \( g \) are surjective, with \( g : A \to B, \ f : B \to C \). Then \( f \circ g : A \to C \). For any \( c \in C \), by surjectivity of \( f \), there exists \( b \in B \) such that \( f(b) = c \). By surjectivity of \( g \), there exists \( a \in A \) such that \( g(a) = b \). Then \( f \circ g(a) = f(g(a)) = f(b) = c \). So, \( f \circ g \) is surjective.
Inverse Relations

An inverse function “undoes” the original function. Example: \( \sqrt[3]{x^3} = x \), so \( f(x) = \sqrt[3]{x} \) is the inverse of \( g(x) = x^3 \).
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**Inverse Relation**

Given a relation $A$ from $A$ to $B$, the **inverse relation** from $B$ to $A$ is defined as

$$R^{-1} = \{(y, x) : (x, y) \in R\}$$
Inverse Relations

12 Functions
12.1 Functions
12.2 Injective and Surjective Functions
12.3 The Pigeonhole Principle
12.4 Composition
12.5 Inverse Functions
12.6 Image and Preimage

Relation: "is the absolute value of"

\[ R = \{ (2, 2), (-2, 2), (1, 1), (-1, 1) \} \]

Function:
\[ f(x) = |x| \]

\[ f: \{ 1, 2, -1, -2 \} \to \{ 1, 2 \} \]

Theorem 12.3
Let \( f: A \to B \) be a function. Then \( f \) is bijective if and only if the inverse relation \( f^{-1} \) is a function from \( B \) to \( A \).

If \( f \) is not injective, then \( f^{-1} \) is not a function.

If \( f(x) \) is not surjective, then the domain of \( f^{-1} \) is not \( B \).
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In the diagram, the function $f: \{-2, -1, 1, 2\} \rightarrow \{1, 2\}$ is shown, where $f(x) = |x|$. The inverse relation $f^{-1}$ is also represented in the diagram.
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*Not a function*
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Identity Function

Given a set $A$, the **identity function** on $A$ is the function that maps every element to itself. That is, $i_A(x) = x$ for every $x \in A$.

Inverse Function

Let $f : A \rightarrow B$ be a bijective function. Note then $f$ is also a relation. Then the inverse relation $f^{-1} : B \rightarrow A$ is the **inverse function** to $f$. 
Inverse Functions

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Under this definition, functions $f$ and $f^{-1}$ have the properties that

$$f^{-1} \circ f = i_A \text{ and } f \circ f^{-1} = i_B$$

That is, $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$. 
Inverse Functions

\[ f : (\mathbb{R} - \{2\}) \rightarrow (\mathbb{R} - \{3\}), \quad f(x) = \frac{3x + 1}{x - 2}. \]

Find \( f^{-1}(x) \).
Inverse Functions

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We want to swap the role of input and output, so we swap \( x \) and \( y \), and we solve for \( y \).
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\[
\begin{align*}
y &= \frac{3x + 1}{x - 2} \\
x &= \frac{3y + 1}{y - 2} \\
yx - 2x &= 3y + 1 \\
yx - 3y &= 2x + 1 \\
y(x - 3) &= 2x + 1 \\
y &= \frac{2x + 1}{x - 3}
\end{align*}
\]

\[ f^{-1}(x) = \frac{2x + 1}{x - 3} \]
Inverse Functions

\[ f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f(x, y) = (y(x^2 + 1), x^3) \] is bijective. Find \( f^{-1}(x) \).
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We want to swap the role of input and output:

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\begin{align*}
    f^{-1}(x, y) &= (A, B) \\
    f(A, B) &= (x, y) \\
    (B(A^2 + 1), A^3) &= (x, y)
\end{align*}
\]

What are \( A, B \) in terms of \( x, y \)?
Inverse Functions

\[ f : \mathbb{R}^2 \to \mathbb{R}^2, \ f(x, y) = (y(x^2 + 1), x^3) \text{ is bijective.} \]

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We want to swap the role of input and output:

\[ f^{-1}(x, y) = (A, B) \quad \text{What are } A, B \text{ in terms of } x, y? \]

\[ f(A, B) = (x, y) \]

\[ (B(A^2 + 1), A^3) = (x, y) \]

\[
\begin{aligned}
  x &= B(A^2 + 1) \\
  y &= A^3 \\
  A &= y^{1/3} \\
  B &= \frac{x}{A^2 + 1} = \frac{x}{y^{2/3} + 1}
\end{aligned}
\]

\[ f^{-1}(x, y) = \left( y^{1/3}, \frac{x}{y^{2/3} + 1} \right) \]
Suppose $f : A \rightarrow B$ is a function, $X \subseteq A$, $Y \subseteq B$.

- We write $f(X) = \{f(x) : x \in X\}$ for the **image** of $X$ under $f$.
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**Example:**

- $f(3) = 9$
- $f(\{3, -3\}) = \{9\}$
- $f([-4, 4]) = [0, 16]$
- $f^{-1}([0, 4]) = [-2, 2]$
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### Notation

#### Image, Preimage

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### Theorem 12.4

Suppose $f : A \rightarrow B$ is a function. Let $W, X \subseteq A$ and $Y, Z \subseteq B$. Then:

- $f(W \cap X) \subseteq f(W) \cap f(X)$
- $f(W \cup X) = f(W) \cup f(X)$
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#### Claim: $f(W \cap X) \subseteq f(W) \cap f(X)$.

#### Proof:

Let $y \in f(W \cap X)$. Then there exists $x \in W \cap X$ such that $f(x) = y$. Since $x$ is in both $W$ and $X$, we know $f(x)$ is in both $f(W)$ and $f(X)$. Therefore, $x \in f(W) \cap f(X)$, and we conclude $f(W \cap X) \subseteq f(W) \cap f(X)$. 

Give an example where $f(W \cap X) \neq f(W) \cap f(X)$.

$f(x) = x^2$, $W = [-1, 0]$, $Z = [0, 1]$. 

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Suppose \( f \) is injective. Then show that \( f(W \cap X) = f(W) \cap f(X) \).
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Suppose $f$ is injective. Then show that $f(W \cap X) = f(W) \cap f(X)$.
Proof: We already showed $f(W \cap X) \subseteq f(W) \cap f(X)$. Let $y \in f(W) \cap f(X)$. Then there exist $x_1$ and $x_2$, $x_1 \in W$ and $x_2 \in X$, such that $f(x_1) = y$ and $f(x_2) = y$. Since $f$ is injective, $x_1 = x_2$, so $x_1 \in W \cap X$, so $y \in f(W \cap X)$. Hence $f(W) \cap f(X) \subseteq f(W \cap X)$, and so the claim holds.