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Suppose A and B are sets. A **function** f from A to B (denoted as $f : A \to B$) is a relation $f \subseteq A \times B$, satisfying the property that for each $a \in A$ the relation f contains exactly one ordered pair of the form (a, b). (Functions pass the vertical line test.)

The statement $(a, b) \in f$ is abbreviated $f(a) = b$.

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Not a function: $f(1)$ has two values.

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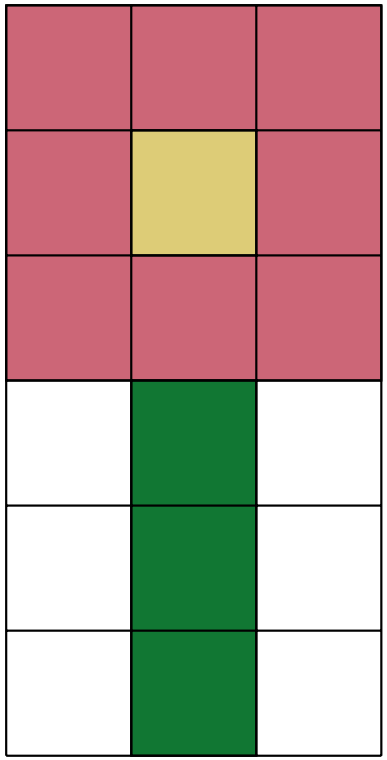
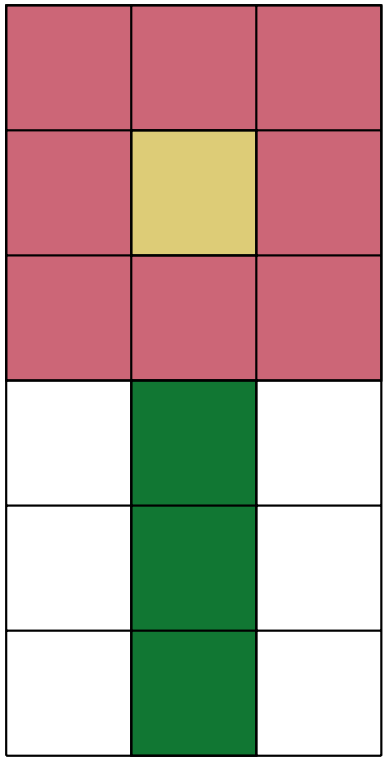
The range is a subset of the codomain.

Functions that aren't from the reals to the reals



Functions that aren't from the reals to the reals

$$p : (\text{location}) \to (\text{color})$$



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Equality in Functions

Equality

Two functions $f : A \to B$ and $g : X \to Y$ are **equal** if (and only if):

- $A = X$, and
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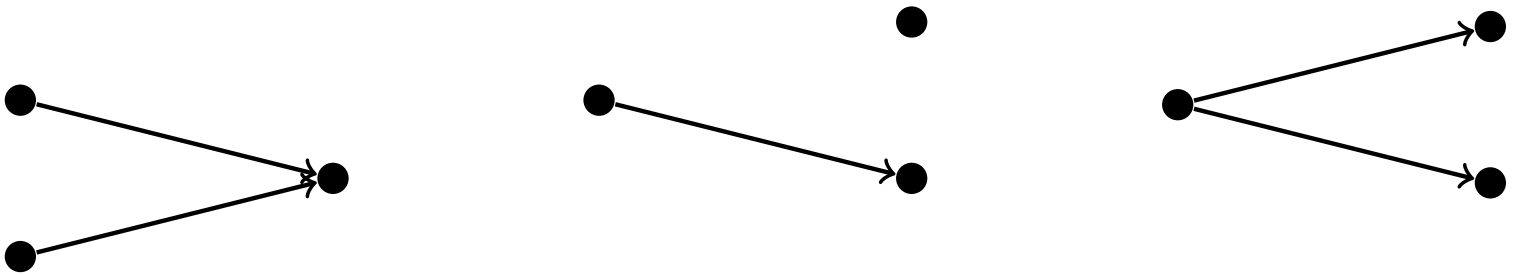

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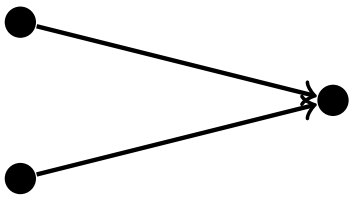
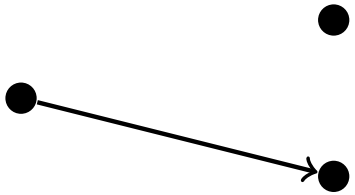
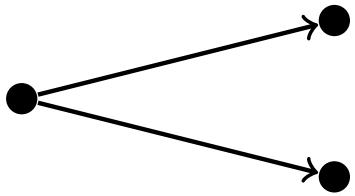


not injective





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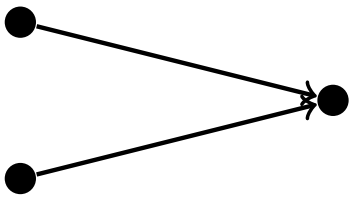

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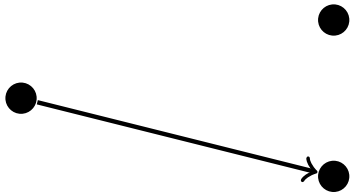
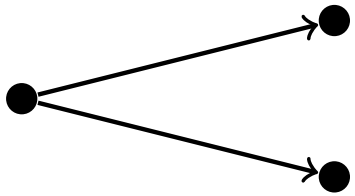


not surjective



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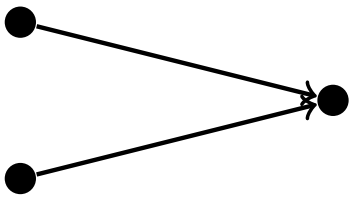

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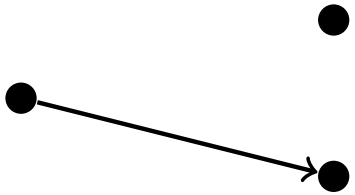

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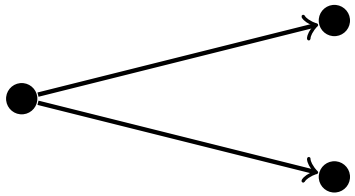

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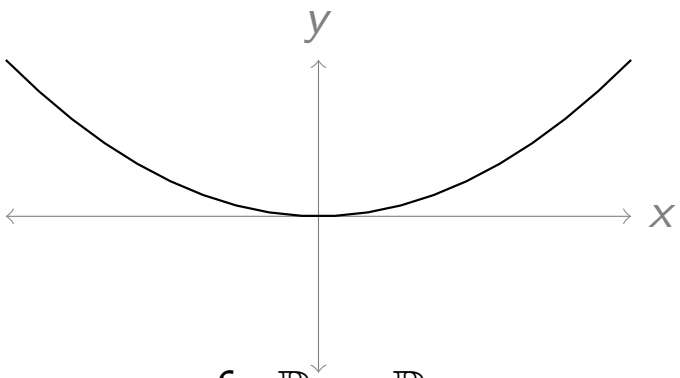

$f : \mathbb{R} \to \mathbb{R}$
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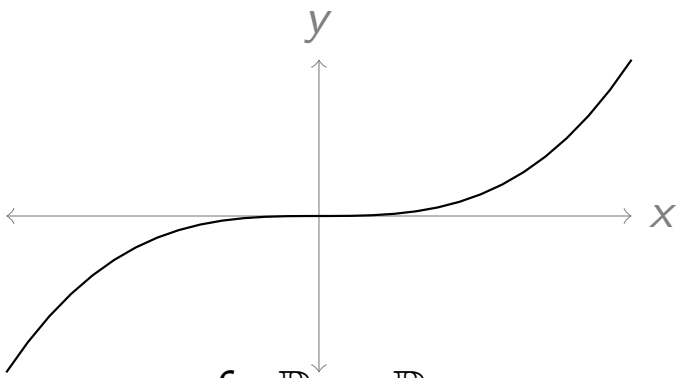

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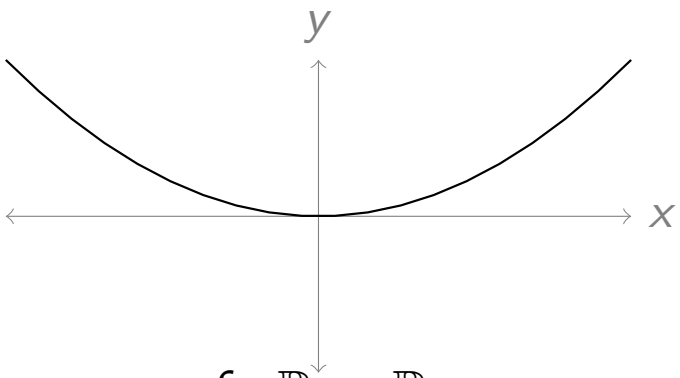

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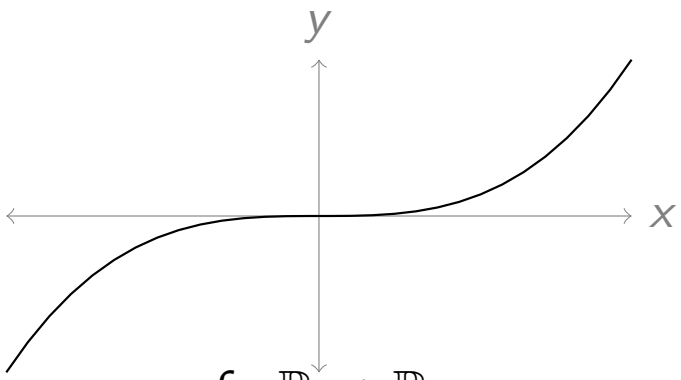
neither



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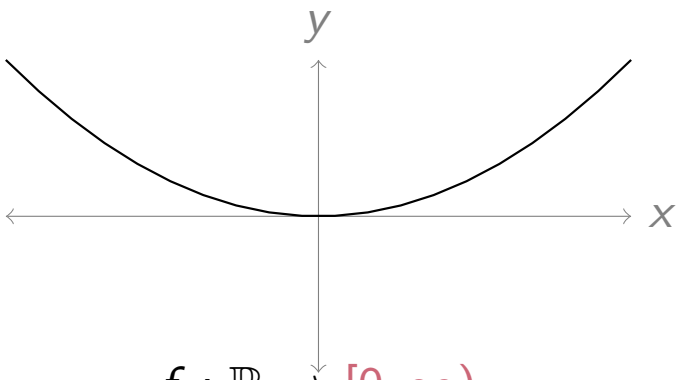

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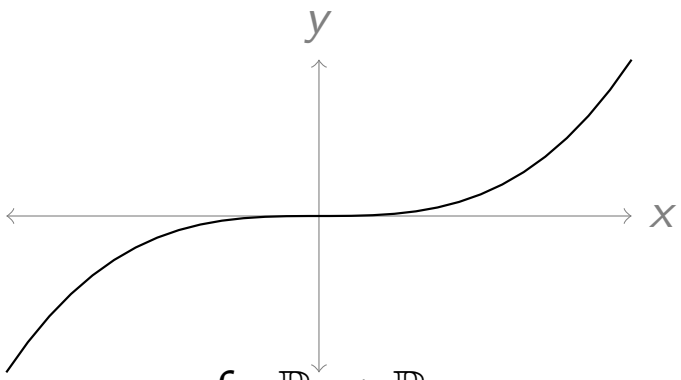

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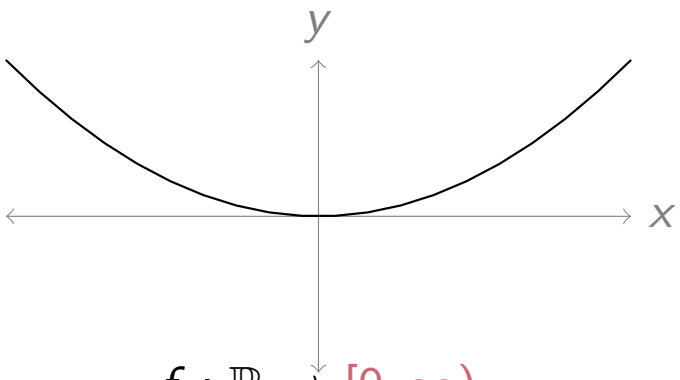

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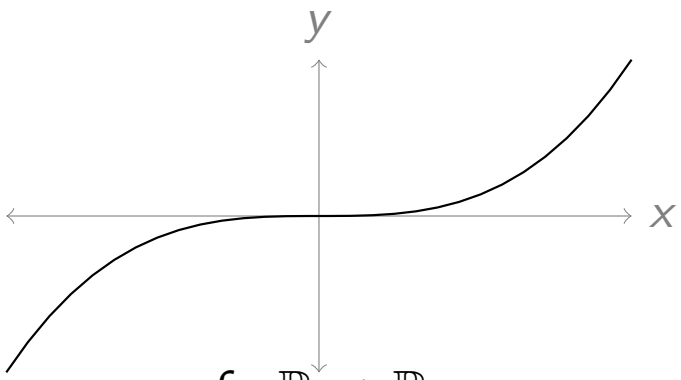

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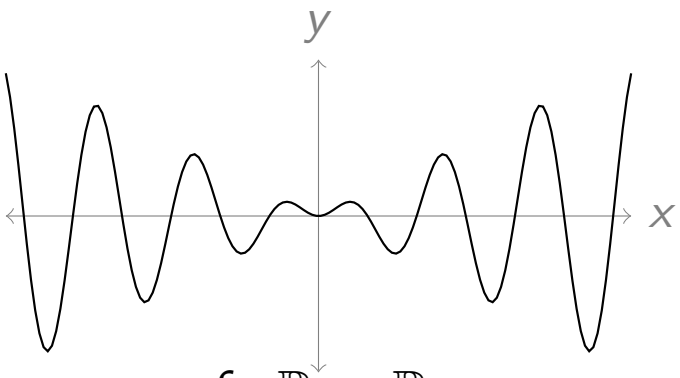

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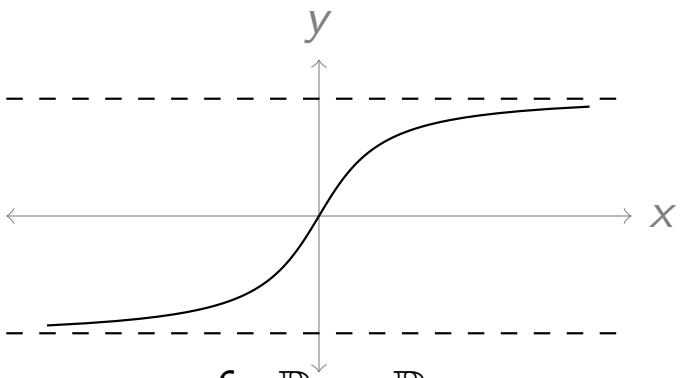

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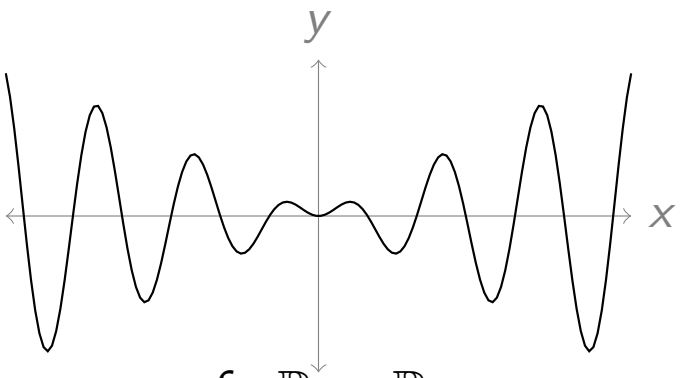

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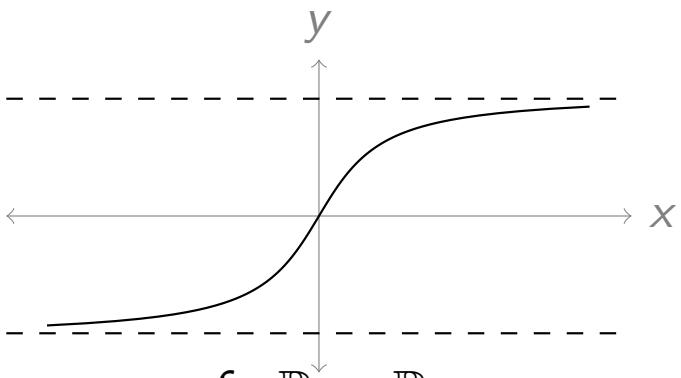

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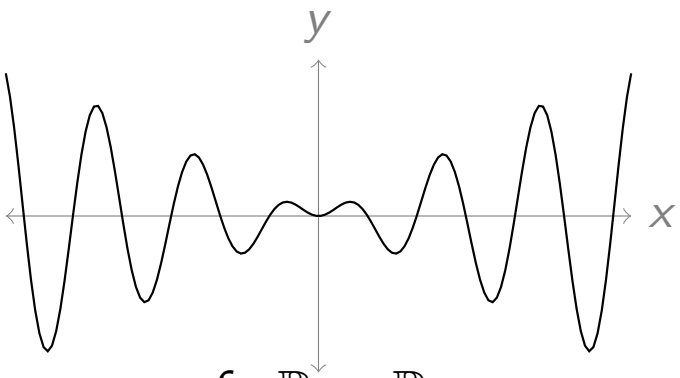

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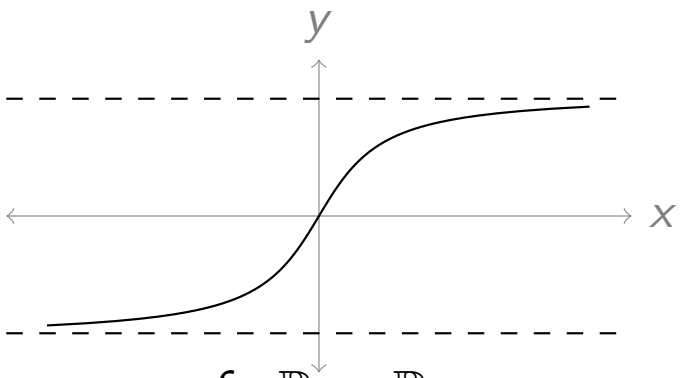

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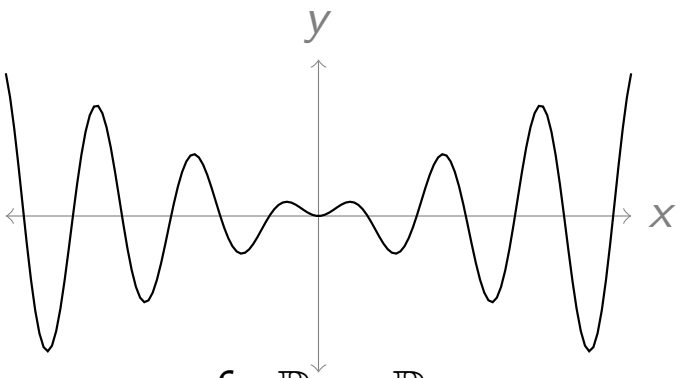

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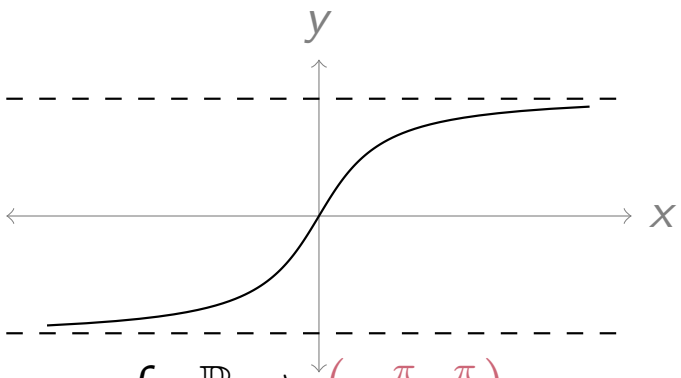

$f : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$f(x) = \arctan x$

injective

surjective

Proving Injectivity and Surjectivity

Prove or disprove:
The function $f(x) = 3x^4 + 1$, $f : \mathbb{R} \to \mathbb{R}$, is injective.

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We notice 1 and -1 are in $\mathbb{R}$, $1 \neq -1$, and $f(1) = f(-1) = 4$.
This shows that f is not injective.

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The function $f(x) = 3x^4 + 1$, $f : \mathbb{R} \to \mathbb{R}$, is surjective.

We notice 0 is in $\mathbb{R}$, but no real value of x gives $f(x) = 0$.
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Proving Injectivity and Surjectivity

Prove that the function

$$f(x) = \frac{x+1}{x-1}, \quad f : (\mathbb{R} - \{1\}) \to (\mathbb{R} - \{1\})$$

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That is:

$$\frac{x+1}{x-1} = \frac{y+1}{y-1}$$

So: $$(x+1)(y-1) = (x-1)(y+1)$$

So: $$xy - x + y - 1 = xy + x - y - 1$$

So: $$-x + y = x - y$$

So: $$2y = 2x$$

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We note that, since $y \neq 1$, $a \in \mathbb{R}$. Furthermore, since the equation $\frac{y+1}{y-1} = 1$ is equivalent to the equation $y + 1 = y - 1$, and so $1 = -1$, which has no solutions, we see that $a \neq 1$. We note further:

$$f(a) = \frac{a+1}{a-1} = \frac{\frac{y+1}{y-1} + 1}{\frac{y+1}{y-1} - 1} = \frac{\frac{y+1+y-1}{y-1}}{\frac{y+1-(y-1)}{y-1}} = \frac{y+1+y-1}{y+1-(y-1)} = \frac{2y}{2} = y$$

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Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

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Is f injective? Is f surjective? Prove your answers.

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f is also surjective. Given any $(a, b) \in \mathbb{R}^2$, $\left(\frac{a+b}{2}, \frac{a-b}{2}\right) \in \mathbb{R}^2$, and $f\left(\frac{a+b}{2}, \frac{a-b}{2}\right) = (a, b)$.

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f is not surjective. We note $(0, 1) \in \mathbb{R}^2$. If $f(x, y) = (0, 1)$, then $x - y = 0$, so $x = y$. But then $x^2 - y^2 = 0$, so $f(x, y) = (0, 0) \neq (0, 1)$.

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Suppose p pigeons are sitting in h pigeonholes.

- If $p > h$, then some hole has more than one pigeon in it.
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Socks and Pigeons

Suppose you have a drawer full of red, blue, and green socks. You're grabbing clothes in the dark. How many socks do you have to take before you're guaranteed to have a matching pair?

You roll a handful of dice all at once. You are guaranteed to have three dice showing the same number. How many dice did you roll?

Pigeonhole Principle

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Suppose everyone in Canada has between 0 and 999,999 hairs on their head. The population of Canada is more than 30 million. What does the Pigeonhole Principle tell us about this situation?

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This is a *non-constructive* proof.

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Notice every number from the original set of ten appears in one of the holes. By picking six numbers, each of which fits into one of the 5 holes, you must have chosen at least two numbers from the same hole. Their sum is 11.

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Suppose you have a pack of cards with the face cards removed. That is, you have 40 cards: four of each suit ($\diamondsuit, \heartsuit, \clubsuit, \spadesuit$), with face values 1-10. What is the minimum number of cards you can draw, and still be guaranteed that there are two cards **of the same suit** that add to 11?

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Example: if $a = 2$, then $2^4 - 2^1 = 16 - 2 = 14 = 2 \times 7$.

Pigeonhole Principle

Proposition: For any $a \in \mathbb{N}$, there exist distinct $k, \ell \in \mathbb{N}$ such that $a^k - a^\ell$ is divisible by 7.

Example: if $a = 2$, then $2^4 - 2^1 = 16 - 2 = 14 = 2 \times 7$.
Example: if $a = 3$, then $3^7 - 3^1 = 2187 - 3 = 2184 = 312 \times 7$

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Proof: There are seven congruence classes mod 7. For every $x \in \mathbb{N}$, a^x is in one of them. Since there are more than seven values of a^x when we let $x \in \mathbb{N}$, two of them must be in the same congruence class mod 7. Then their difference is divisible by 7.

Actually: there must exist k, ℓ in $\{1, \ldots, 8\}$ such that $a^k - a^\ell$ is divisible by 7.

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Proof We will use the pigeonhole principle. Let the natural numbers be partitioned into three "holes" by their equivalence class mod 5, as follows:

$$[0], \quad [1] \cup [4], \quad [2] \cup [3]$$

By the pigeonhole principle, if we pick four natural numbers, two of them must be in the same hole. If they are in the same equivalence class mod 5, then their difference is divisible by 5. If they are in different equivalence classes (but the same hole), then their sum is divisible by 5.

Function Composition

Function Composition

Let f and g be functions, $f : B \to C$ and $g : A \to B$. Then we defined the **composition** of f and g to be

$$(f \circ g)(x) = f(g(x))$$

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Suppose $f : \mathbb{R}^2 \to \mathbb{R}^3$, $g : \mathbb{R} \to \mathbb{R}^2$, with
$f(x, y) = (x^2, x + y, y^2)$, and $g(x) = (3x, 4x)$.
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$f \circ g(x) = f(3x, 4x) = (9x^2, 7x, 16x^2)$

Composition and Injectivity, Surjectivity

For the following characteristics, give an example of functions f and g that have them, or show that none exists.
Assume that the range of g is the same as the domain of f.

- f injective, $f \circ g$ not injective.
- f injective, g injective, $f \circ g$ not injective.
- g surjective, $f \circ g$ not surjective.
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  Suppose f and g are both injective. Then if $f(g(x)) = f(g(y))$, by injectivity of f, $g(x) = g(y)$. Then, by injectivity of g, $g(x) = g(y)$ implies $x = y$. So $f \circ g$ is injective.

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- g surjective, $f \circ g$ not surjective.
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- f surjective, g surjective, $f \circ g$ not surjective.
  Suppose f and g are surjective, with $g : A \to B$, $f : B \to C$. Then $f \circ g : A \to C$. For any $c \in C$, by surjectivity of f, there exists $b \in B$ such that $f(b) = c$. By surjectivity of g, there exists $a \in A$ such that $g(a) = b$. Then $f \circ g(a) = f(g(a)) = f(b) = c$. So, $f \circ g$ is surjective.

Inverse Relations

An inverse function "undoes" the original function. Example: $\sqrt[3]{x^3} = x$, so $f(x) = \sqrt[3]{x}$ is the inverse of $g(x) = x^3$.

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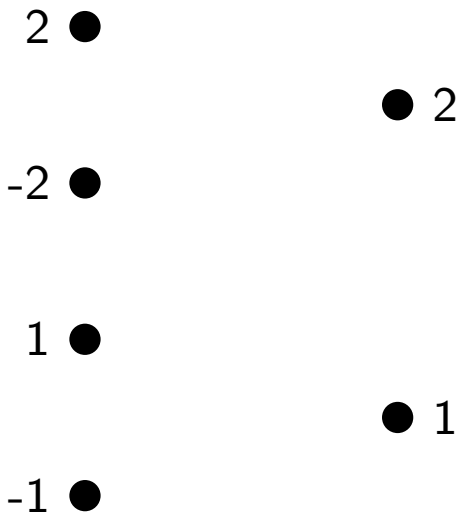
Inverse Relation

Given a relation A from A to B, the **inverse relation** from B to A is defined as

$$R^{-1} = \{(y, x) \ : \ (x, y) \in R\}$$

Inverse Relations



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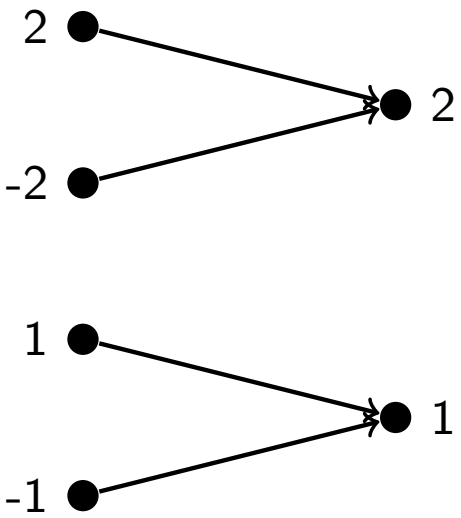

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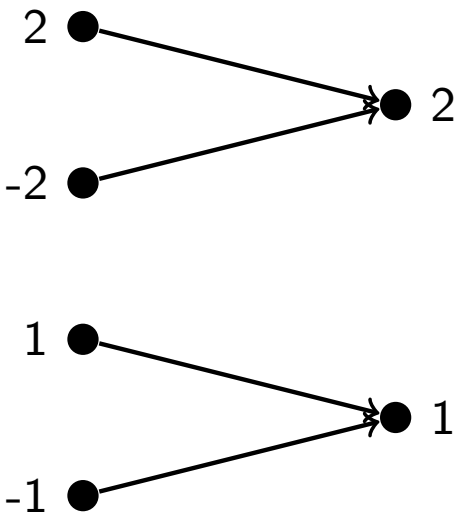
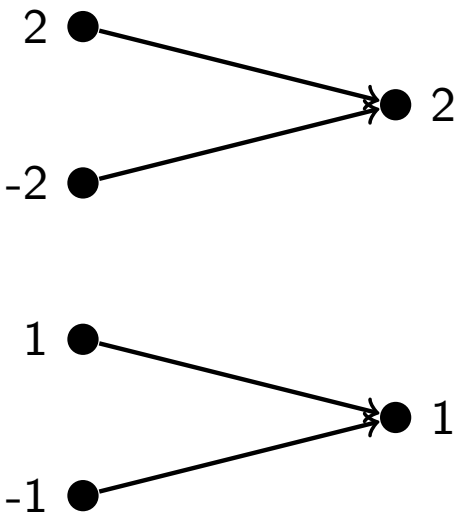


Relation: "is the absolute value of"

$$R = \{(2,2),(-2,2),(1,1),(-1,1)\}$$

Inverse Relations



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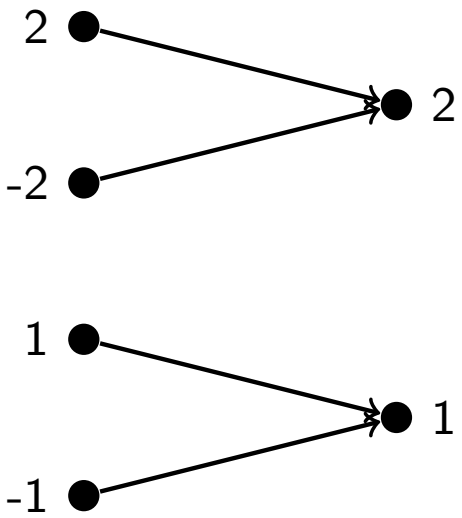

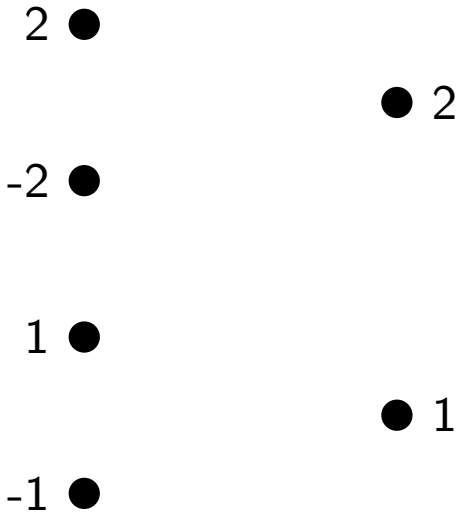

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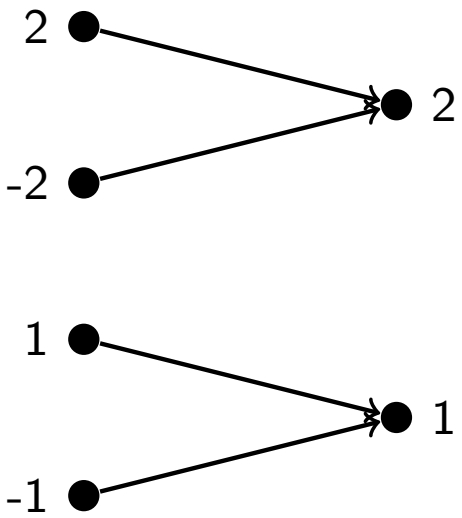

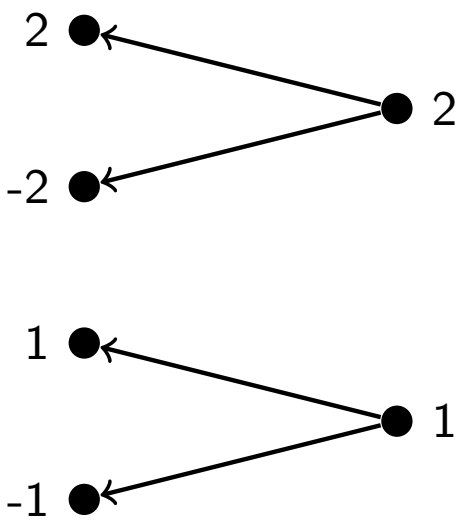

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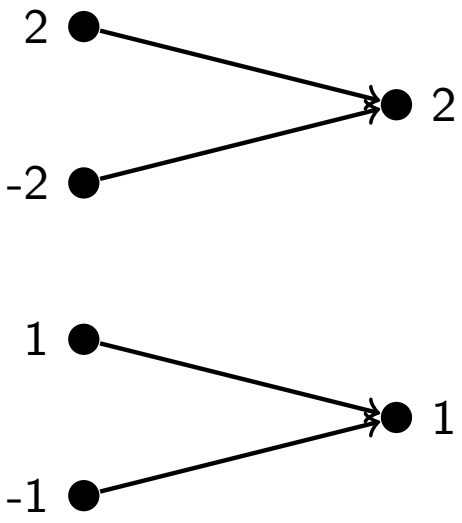

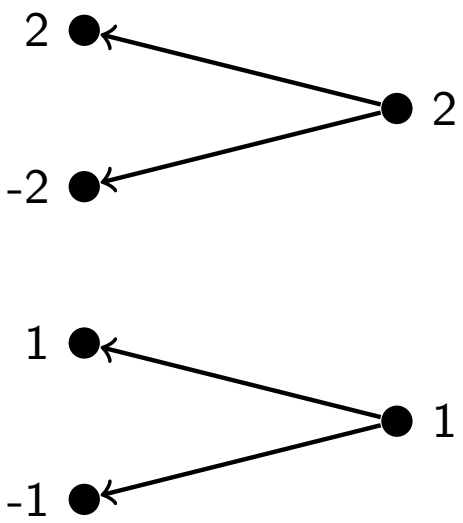

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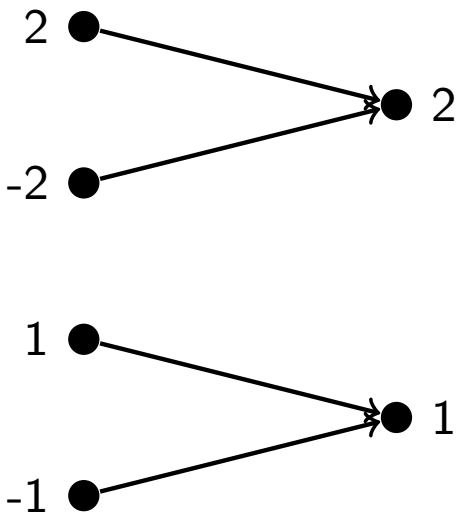

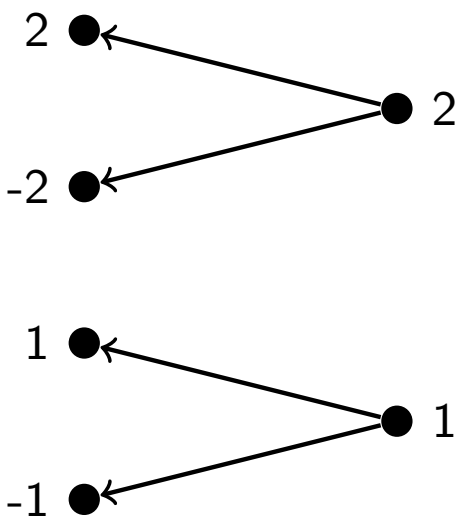

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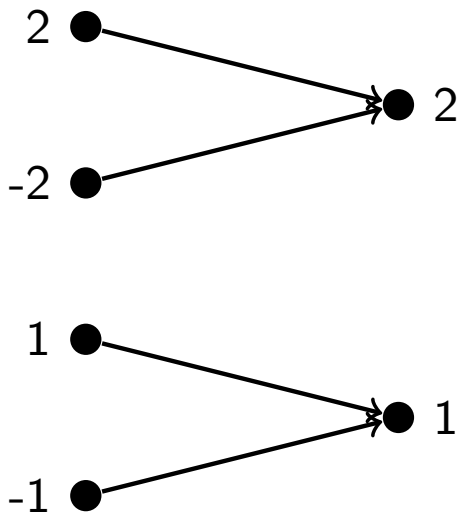

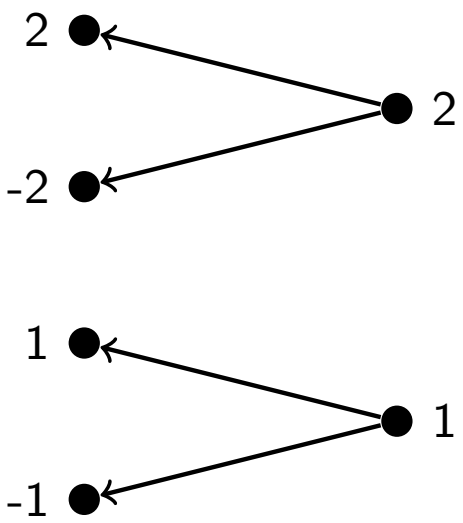

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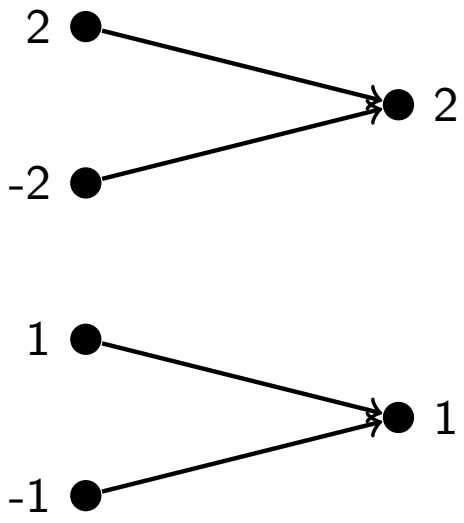

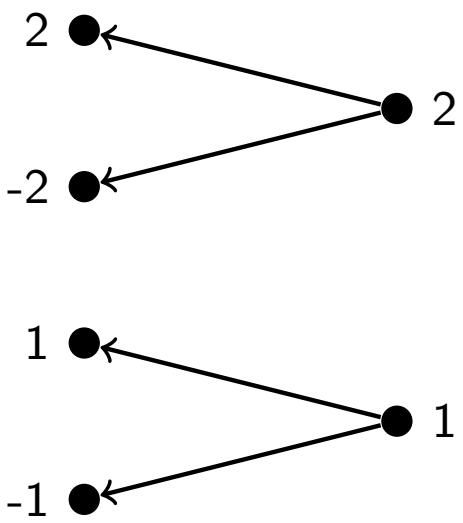

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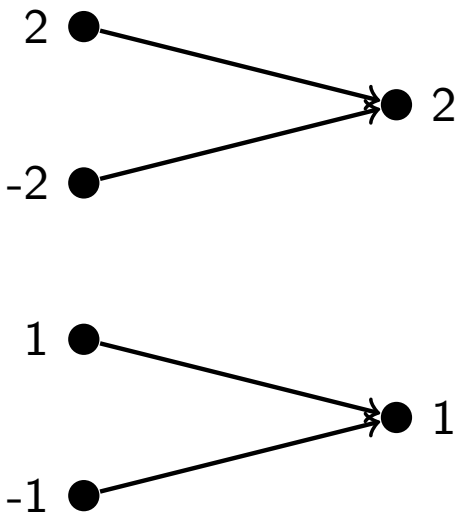

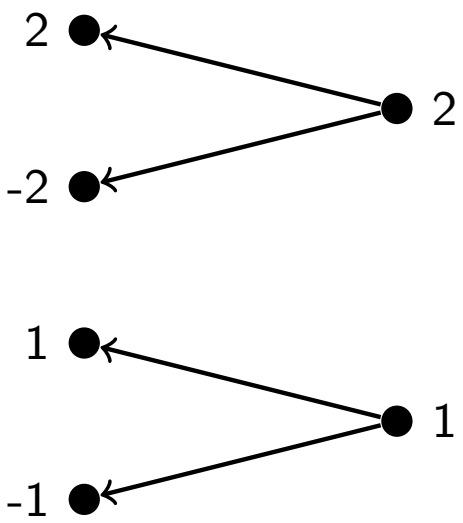

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If f is not injective, then f^{-1} is not a function.

If $f(x)$ is not surjective, then the domain of f^{-1} is not B.

Inverse Functions

Identity Function

Given a set A, the **identity function** on A is the function that maps every element to itself. That is, $i_A(x) = x$ for every $x \in A$.

Inverse Function

Let $f : A \to B$ be a bijective function. Note then f is also a relation. Then the inverse relation $f^{-1} : B \to A$ is the **inverse function** to f.

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Under this definition, functions f and f^{-1} have the properties that

$$f^{-1} \circ f = i_A \text{ and } f \circ f^{-1} = i_B$$

That is, $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Inverse Functions

$f : (\mathbb{R} - \{2\}) \to (\mathbb{R} - \{3\})$, $f(x) = \dfrac{3x+1}{x-2}$.

Find $f^{-1}(x)$.

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$$\begin{aligned} y &= \frac{3x+1}{x-2} \\ x &= \frac{3y+1}{y-2} \\ yx - 2x &= 3y + 1 \\ yx - 3y &= 2x + 1 \\ y(x-3) &= 2x + 1 \\ y &= \frac{2x+1}{x-3} \end{aligned}$$

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$f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (y(x^2 + 1), x^3)$ is bijective.
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$$f^{-1}(x, y) = (A, B) \qquad \text{What are } A,\ B \text{ in terms of } x,\ y?$$
$$f(A, B) = (x, y)$$
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$$\begin{cases} x &= B(A^2 + 1) \\ y &= A^3 \end{cases}$$
$$A = y^{1/3}$$
$$B = \frac{x}{A^2 + 1} = \frac{x}{y^{2/3} + 1}$$

$$f^{-1}(x, y) = \left(y^{1/3}, \frac{x}{y^{2/3} + 1}\right)$$

Notation

Image, Preimage

Suppose $f : A \to B$ is a function, $X \subseteq A$, $Y \subseteq B$.

- We write $f(X) = \{f(x) : x \in X\}$ for the **image** of X under f.
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Example: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

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- $f([-4, 4]) =$
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Image, Preimage

Suppose $f : A \to B$ is a function, $X \subseteq A$, $Y \subseteq B$.

- We write $f(X) = \{f(x) : x \in X\}$ for the **image** of X under f.
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Example: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

- $f(3) = 9$
- $f(\{3, -3\}) = \{9\}$
- $f([-4, 4]) = [0, 16]$
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Properties

Theorem 12.4

Suppose $f : A \to B$ is a function. Let $W, X \subseteq A$ and $Y, Z \subseteq B$. Then:

- $f(W \cap X) \subseteq f(W) \cap f(X)$
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Give an example where $f(W \cap X) \neq f(W) \cap f(X)$.

$f(x) = x^2$, $W = [-1, 0]$, $Z = [0, 1]$.

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Suppose f is injective. Then show that $f(W \cap X) = f(W) \cap f(X)$.

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Suppose f is injective. Then show that $f(W \cap X) = f(W) \cap f(X)$.
Proof: We already showed $f(W \cap X) \subseteq f(W) \cap f(X)$. Let $y \in f(W) \cap f(X)$. Then there exist x_1 and x_2, $x_1 \in W$ and $x_2 \in X$, such that $f(x_1) = y$ and $f(x_2) = y$. Since f is injective, $x_1 = x_2$, so $x_1 \in W \cap X$, so $y \in f(W \cap X)$. Hence $f(W) \cap f(X) \subseteq f(W \cap X)$, and so the claim holds.