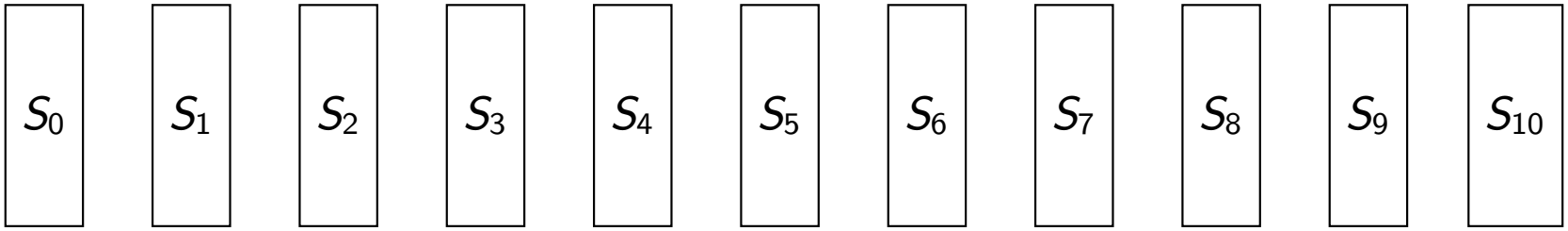
Mathematical Induction

10. Mathematical Induction
10.1 Strong Induction
10.2 Smallest Counterexample



Like dominoes!

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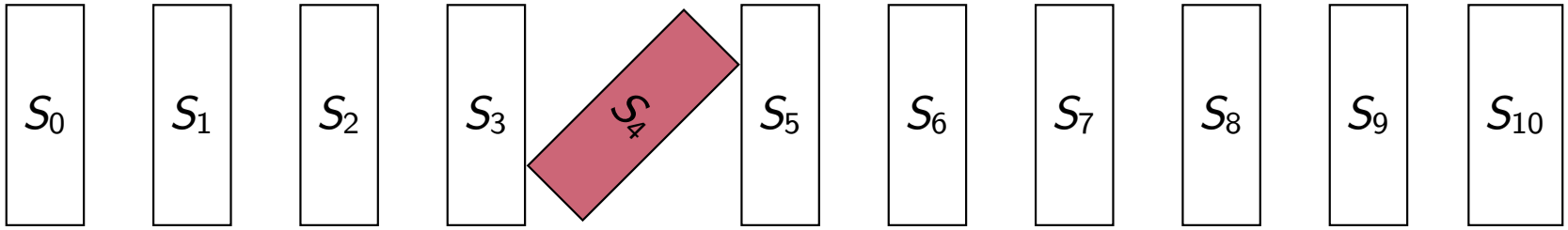

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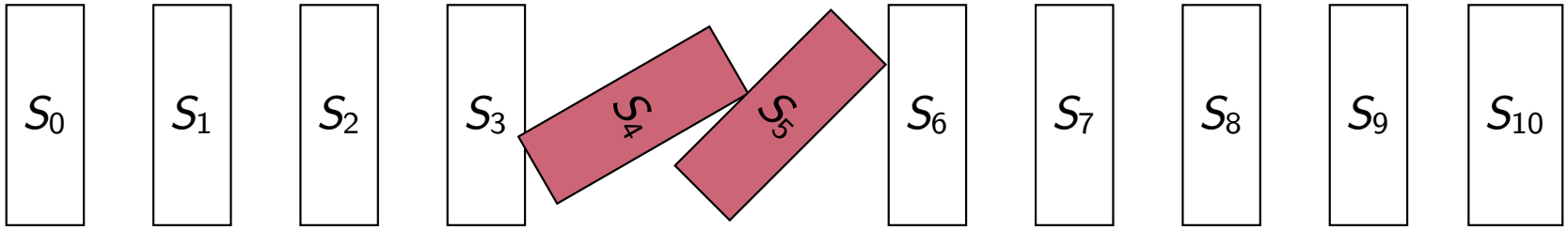

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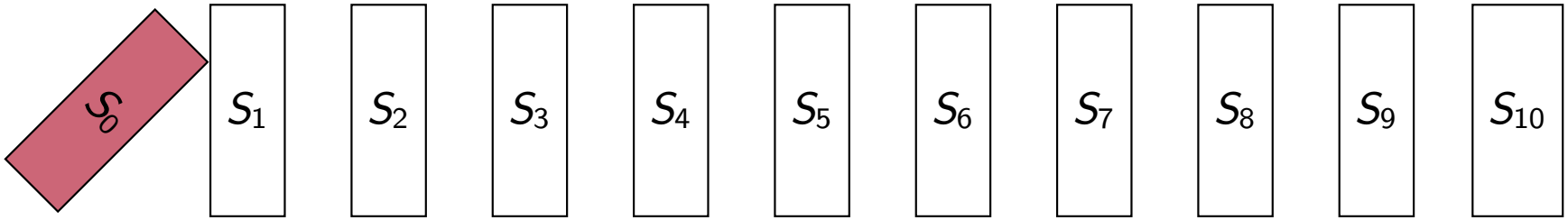

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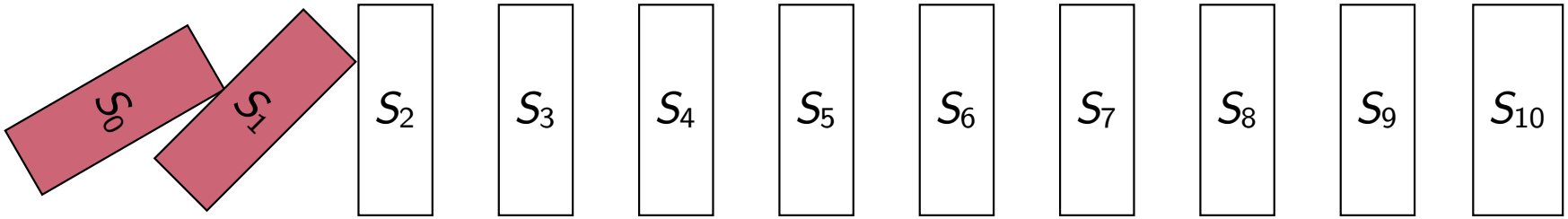

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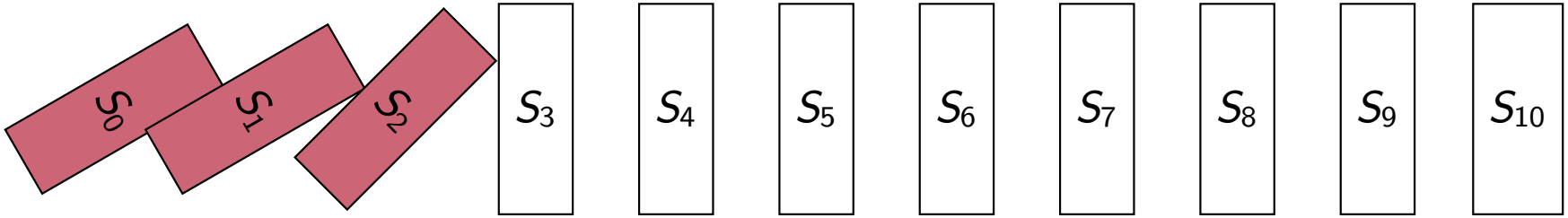

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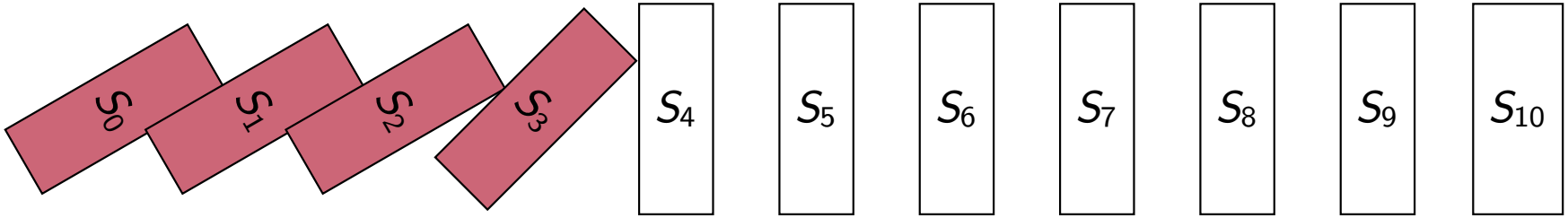

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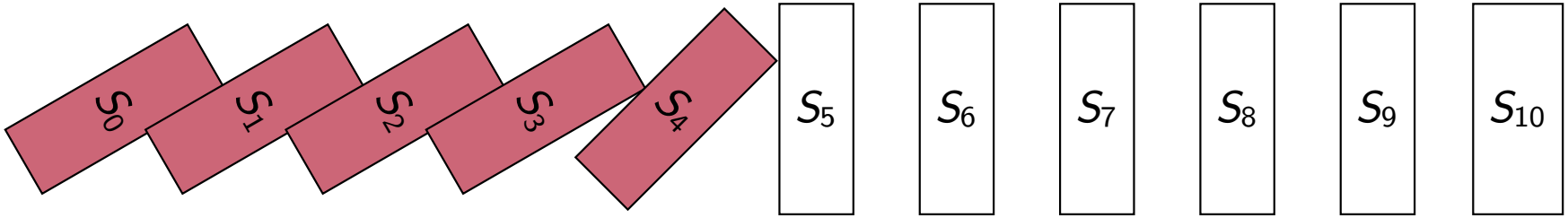

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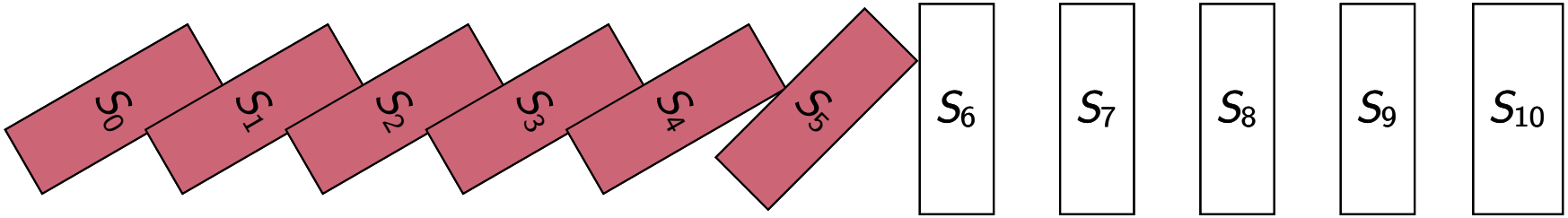

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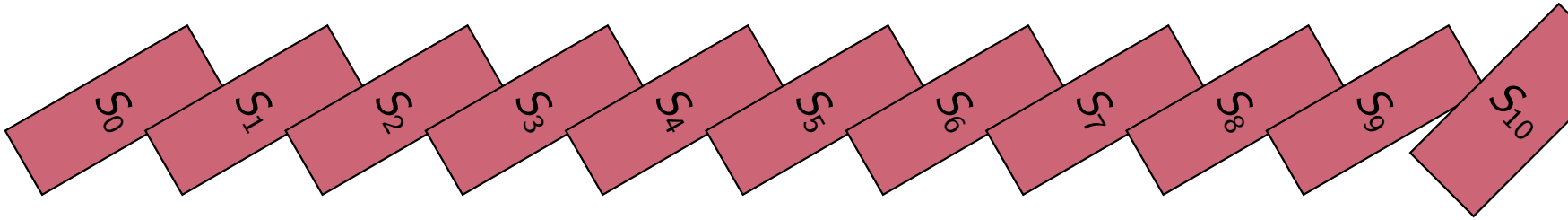

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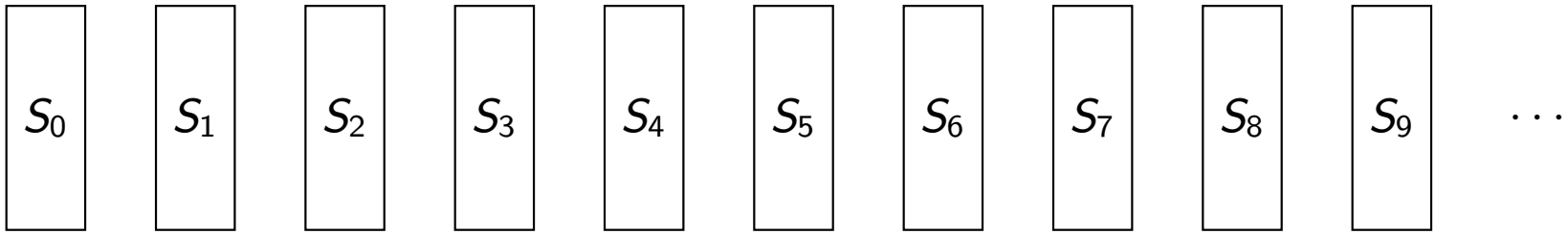

- For any $k \in \mathbb{N}$, if S_k is true, then S_{k+1} is true.
  (This is called the "inductive step." The assumption S_k is true is called the "inductive hypothesis.")

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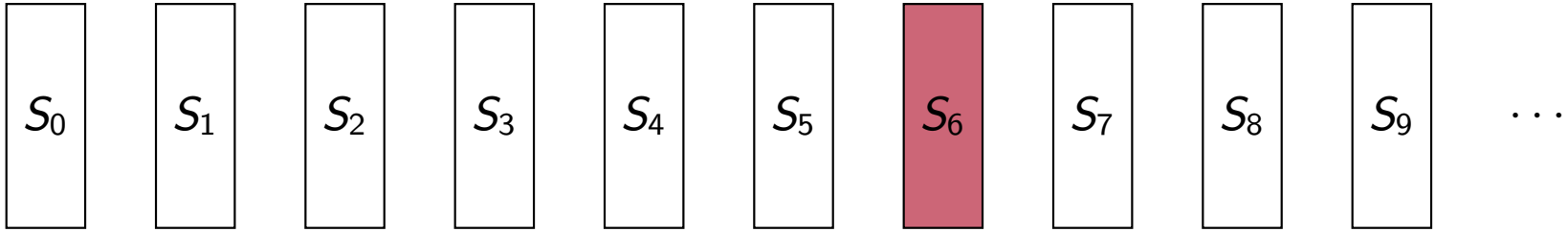

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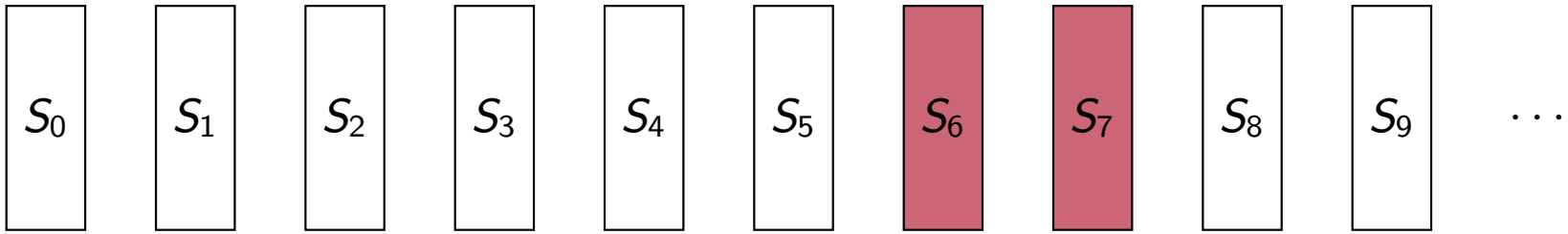

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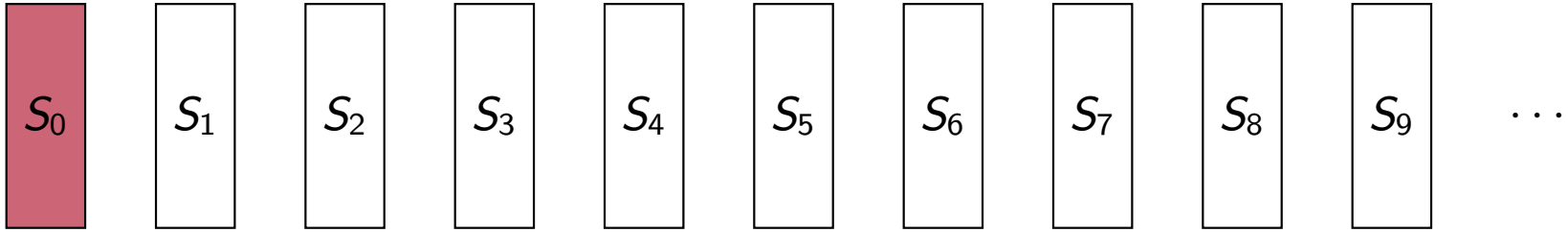

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Mathematical Induction

10. Mathematical Induction
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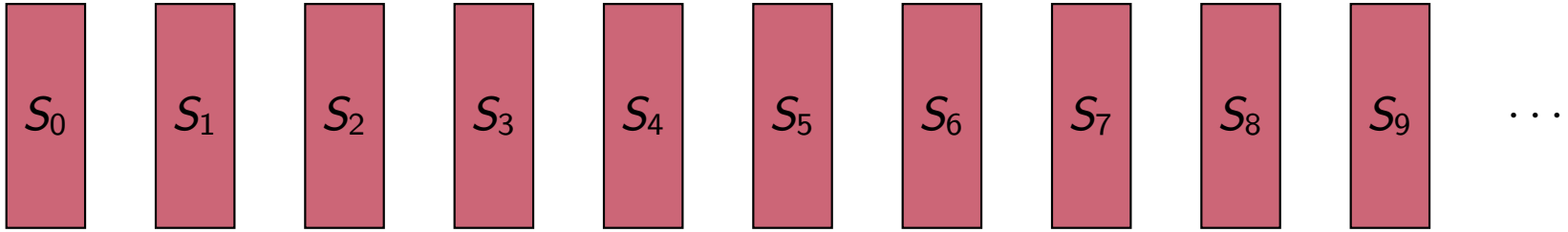

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- Therefore, S_k is true for all $k \geq a$.

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Proposition: Suppose $n \in \mathbb{N}$. Then $5|(n^5 - n)$

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Proof: (by induction)

- Let $k \in \mathbb{N}$, and suppose $5|(k^5 - k)$. Then there exists some $x \in \mathbb{Z}$ such that $k^5 - k = 5x$. Then:

$$\begin{aligned}(k+1)^5 - (k+1) &= [k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1] - (k+1) \\ &= [k^5 - k] + [5k^4 + 10k^3 + 10k^2 + 5k] \\ &= 5x + [5k^4 + 10k^3 + 10k^2 + 5k] \\ &= 5[x + k^4 + 2k^3 + 2k^2 + k]\end{aligned}$$

  So, $5 \left| \left((k+1)^5 - (k+1) \right) \right.$

- When $n = 1$, $n^5 - n = 0$, so $5|(n^5 - n)$.
- It follow by induction that, for every $n \in \mathbb{N}$, $5|(n^5 - n)$.

10. Mathematical Induction
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Proposition: Suppose $n \in \mathbb{N}$. Then $\displaystyle\sum_{k=0}^{n-1} 2^k = 2^n - 1$.

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Proposition: Suppose $n \in \mathbb{N}$. Then $\sum_{k=0}^{n-1} 2^k = 2^n - 1$.

Proof: (by induction)

- Let $m \in \mathbb{N}$, and suppose $\sum_{k=0}^{m-1} 2^k = 2^m - 1$. Then:

$$\begin{aligned}\sum_{k=0}^{m} 2^k &= 2^m + \sum_{k=0}^{m-1} 2^k \\ &= 2^m + (2^m - 1) = 2(2^m) - 1 = 2^{m+1} - 1\end{aligned}$$

  So, if the statement holds for $n = m$, then it also holds for $n = m + 1$.

- When $n = 1$, $\sum_{k=0}^{n-1} 2^k = 2^0 = 1$, and $1 = 2^n - 1$, so the statement holds when $n = 1$.

- It follow by induction that, for every $n \in \mathbb{N}$, $\sum_{k=0}^{n-1} 2^k = 2^n - 1$.

10. Mathematical Induction
10.1 Strong Induction
10.2 Smallest Counterexample

Proposition: Let $f(x) = x \ln x$. Then for all $n \in \mathbb{N}$ such that $n \geq 2$,

$$f^{(n)}(x) = (-1)^n (n-2)! x^{1-n}.$$

(Recall $0! = 1$.)

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(Recall $0! = 1$.)

Proof: (by induction)

- Suppose $k \in \mathbb{N}$ and $f^{(k)}(x) = (-1)^k (k-2)! x^{1-k}$. Then

$$\begin{aligned} f^{(k+1)}(x) &= \frac{d}{dx}\left\{f^{(k)}(x)\right\} \\ &= \frac{d}{dx}\left\{(-1)^k (k-2)! x^{1-k}\right\} \\ &= (1-k)(-1)^k (k-2)! x^{1-k-1} \\ &= -(k-1)(-1)^k (k-2)! x^{1-(k+1)} \\ &= (-1)^{k+1} (k-1)! x^{1-k+1} \end{aligned}$$

  So, if the statement holds for $n = k$, then is also holds for $n = k+1$.

- Note $f'(x) = \ln x + 1$ (by the product rule), so $f^{(2)}(x) = \frac{1}{x} = (-1)^2 0! x^{1-2}$, so the statement holds for $n = 2$.

- By induction, we conclude the statement holds for all natural $n \geq 2$.

Cautionary Tale

10. Mathematical Induction
10.1 Strong Induction
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Proposition: Let $f(x) = e^x$ Then $f^{(n)}(x) = 0$ for all $n \in \mathbb{N}$.

Proof:

- Suppose $f^{(n)}(x) = 0$. Then $f^{(n+1)}(x) = \frac{d}{dx}\{0\} = 0$. So, if the statement holds for n, then it also holds for $n + 1$.
- We conclude by induction that $f^{(n)}(x) = 0$ for all $n \in \mathbb{N}$.

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We need a base case! This is clearly false.

Induction: Triominoes

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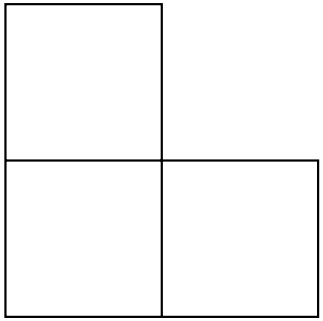

Triomino

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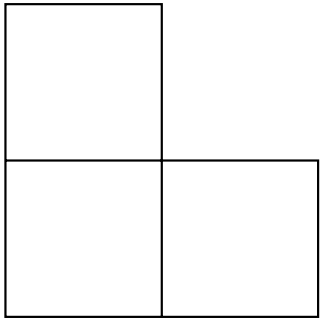

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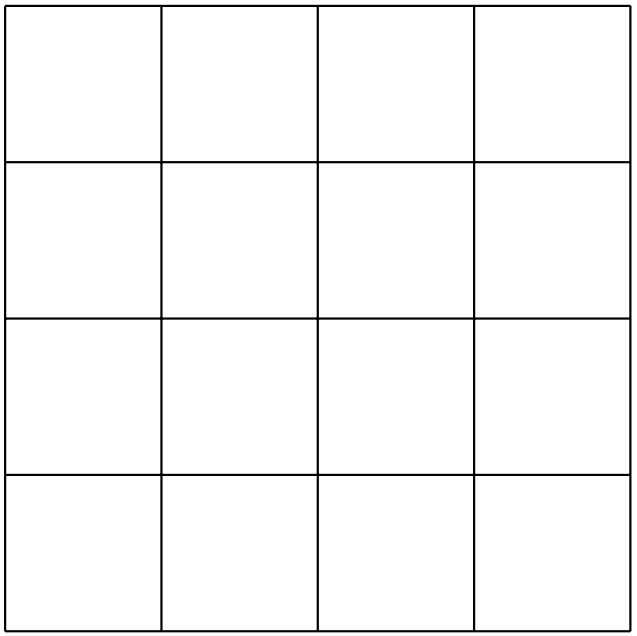

4×4 chessboard

Induction: Triominoes

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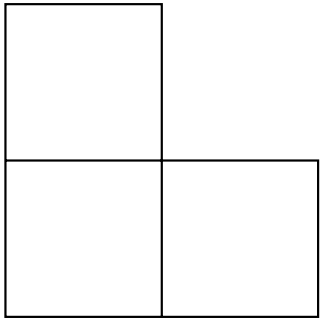

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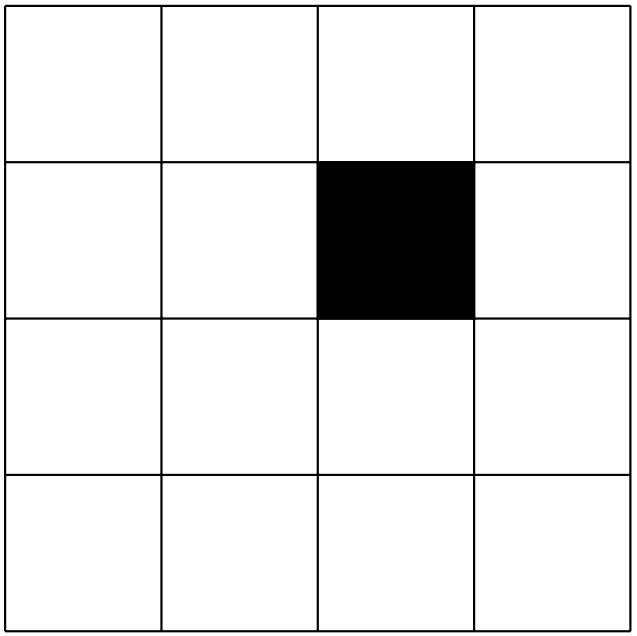

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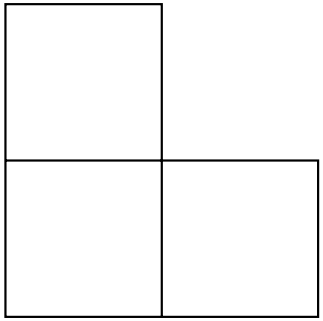

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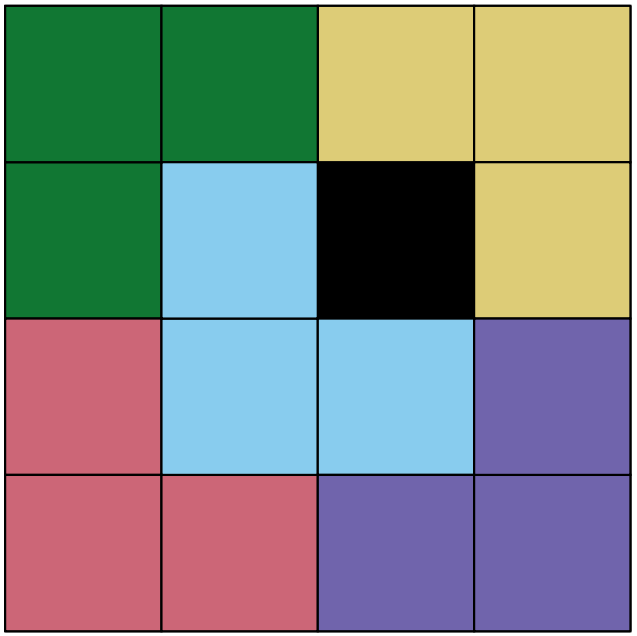

4×4 chessboard

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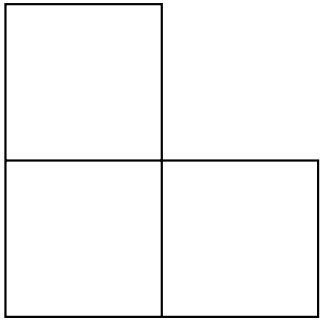
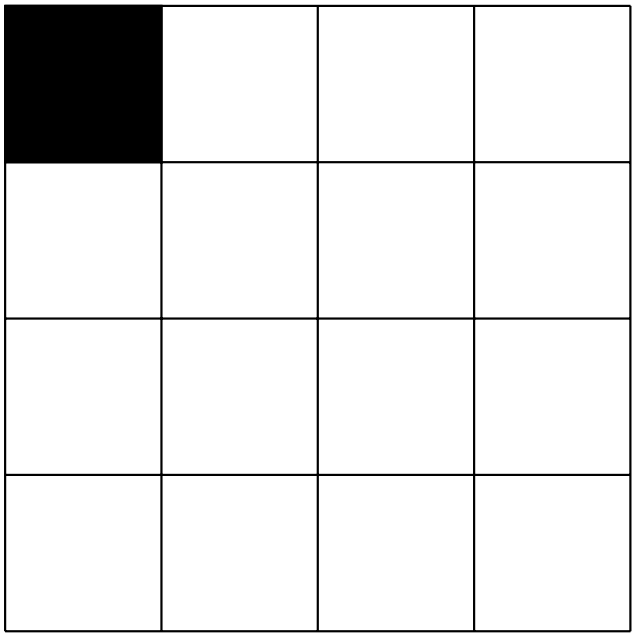


Triomino



4×4 chessboard

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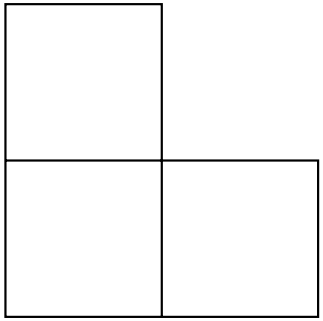

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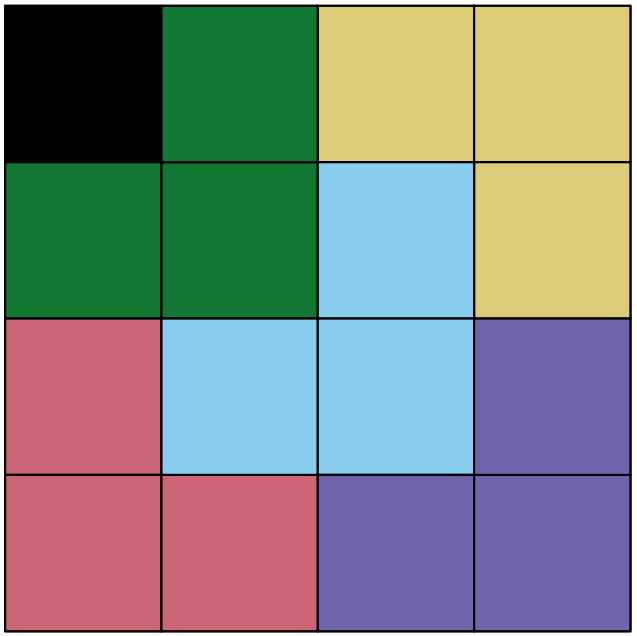

4×4 chessboard

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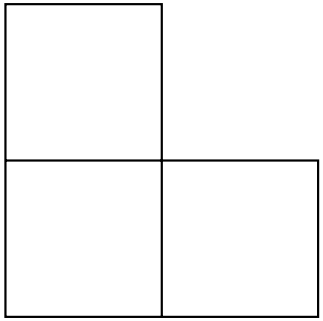
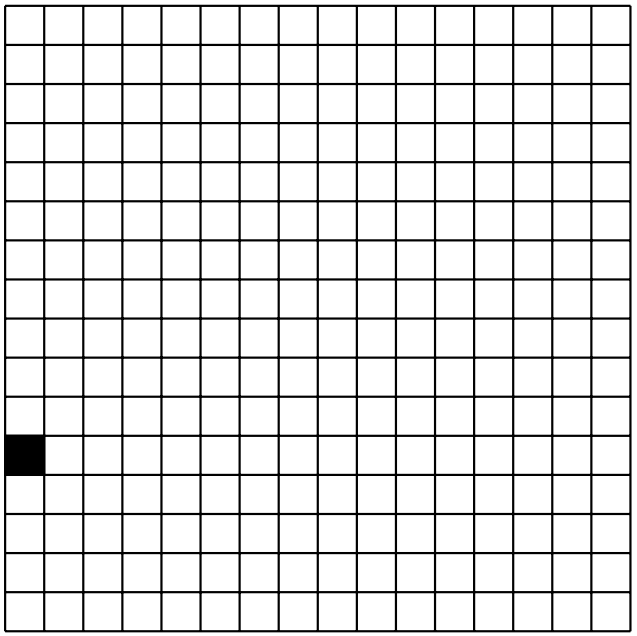


Triomino



Induction: Triominoes

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Proposition: For any $n \in \mathbb{N}$, a chessboard of dimensions $2^n \times 2^n$ that is missing any single square can be tiled by triominoes.

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- When $n = 1$, the board is exactly the shape of a single triomino, so it can be tiled.
- Suppose any board of dimensions $2^n \times 2^n$ missing one square can be tiled. Consider a board of dimension $2^{n+1} \times 2^{n+1}$, missing any one square.

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- Suppose any board of dimensions $2^n \times 2^n$ missing one square can be tiled. Consider a board of dimension $2^{n+1} \times 2^{n+1}$, missing any one square. Then we can cut the board into four equal-sized boards, each of dimension $2^n \times 2^n$. One of these contains the missing square; by the inductive hypothesis, this can be tiled. The other three smaller boards do *not* contain any missing pieces.

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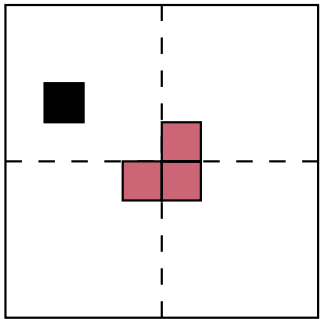
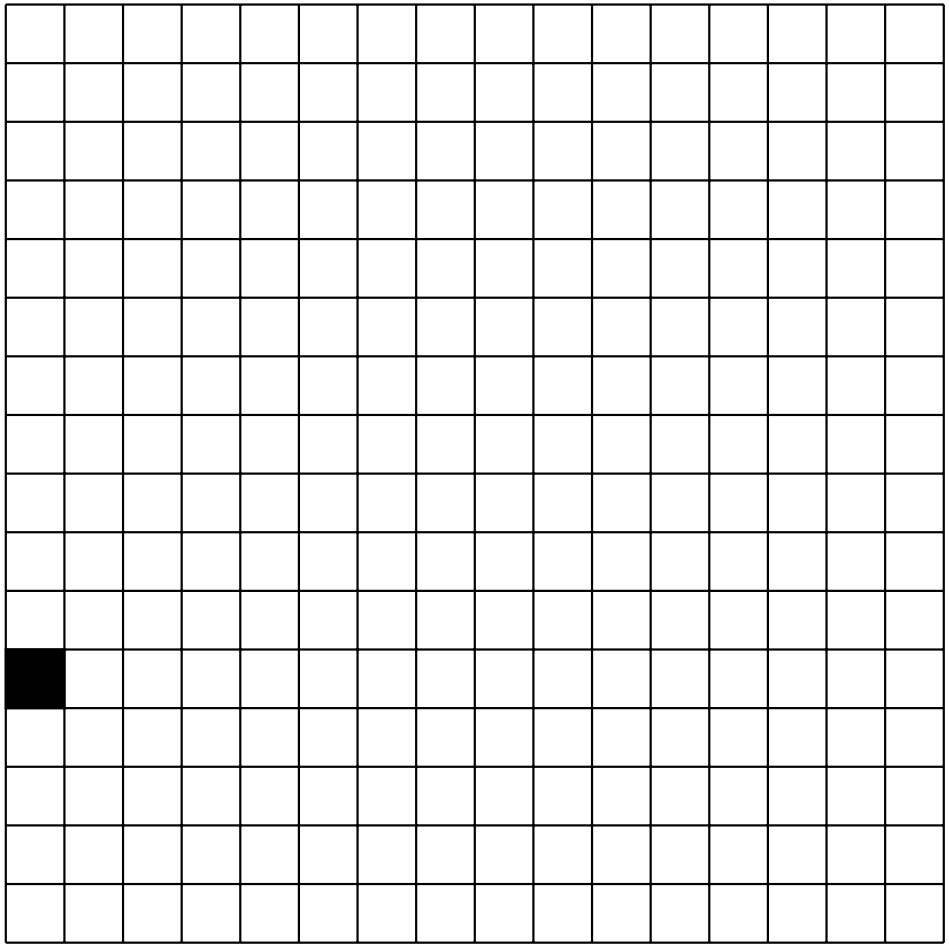
- We conclude that the proposition holds.



Triominoes tiling



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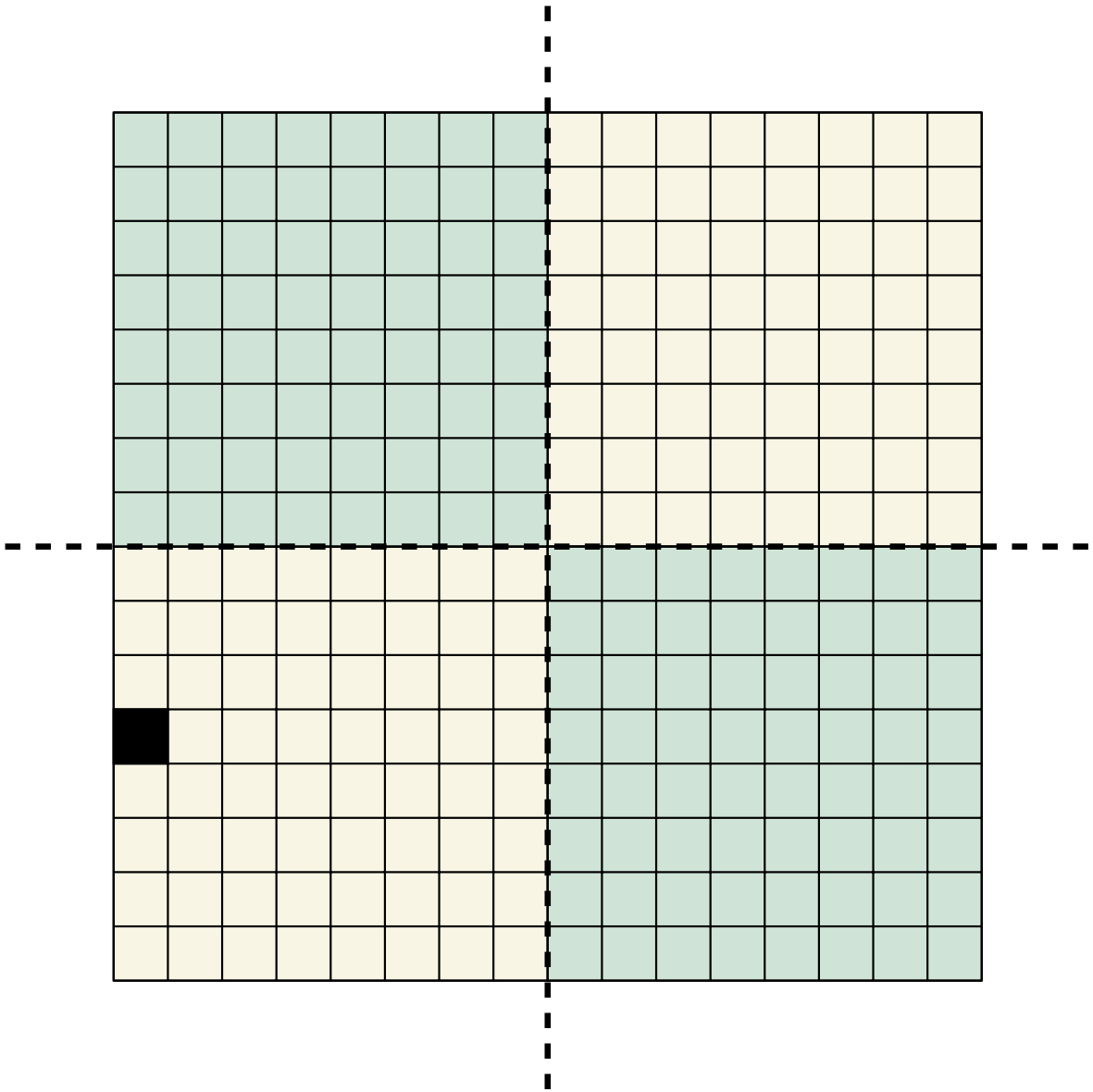

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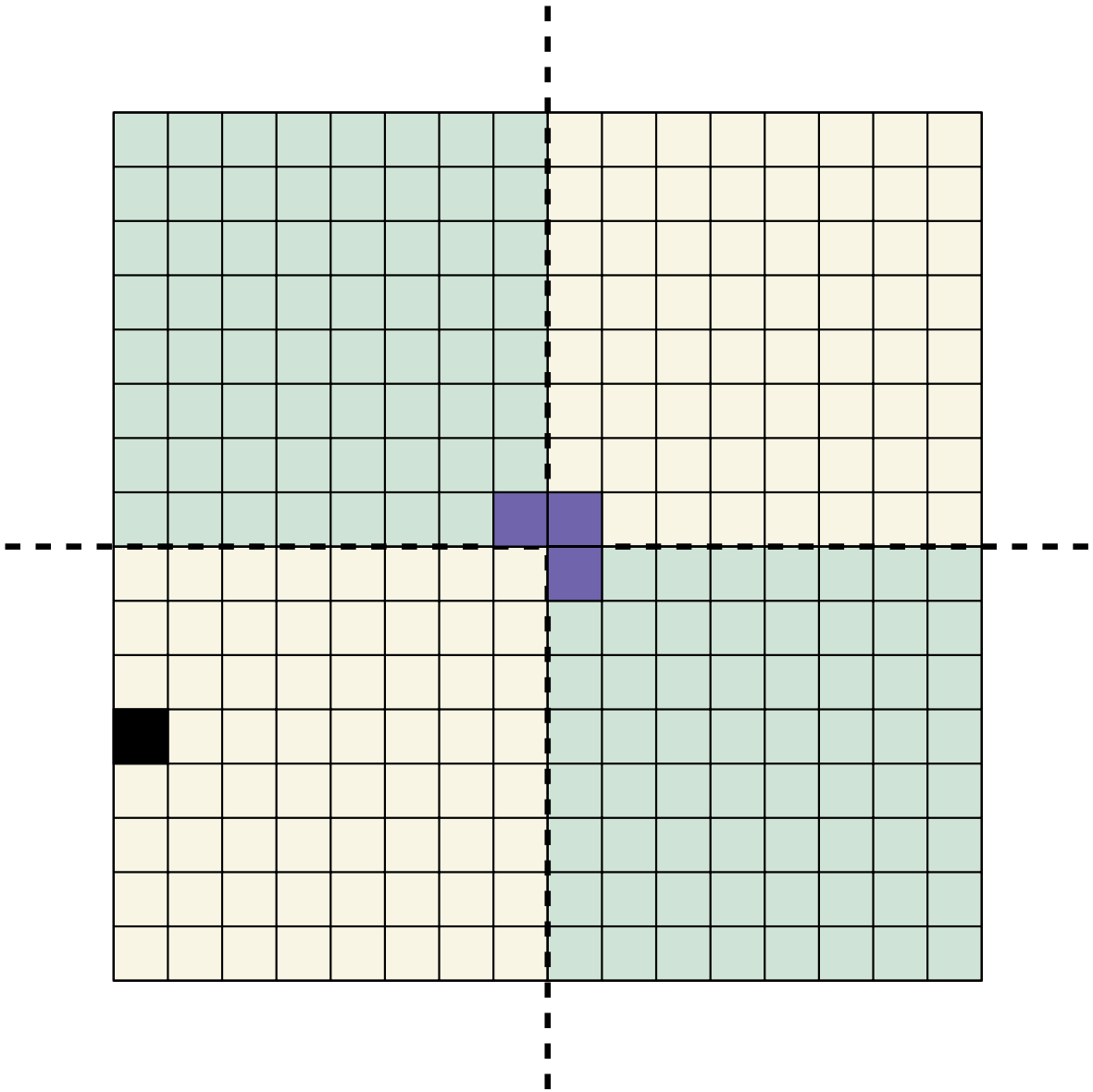
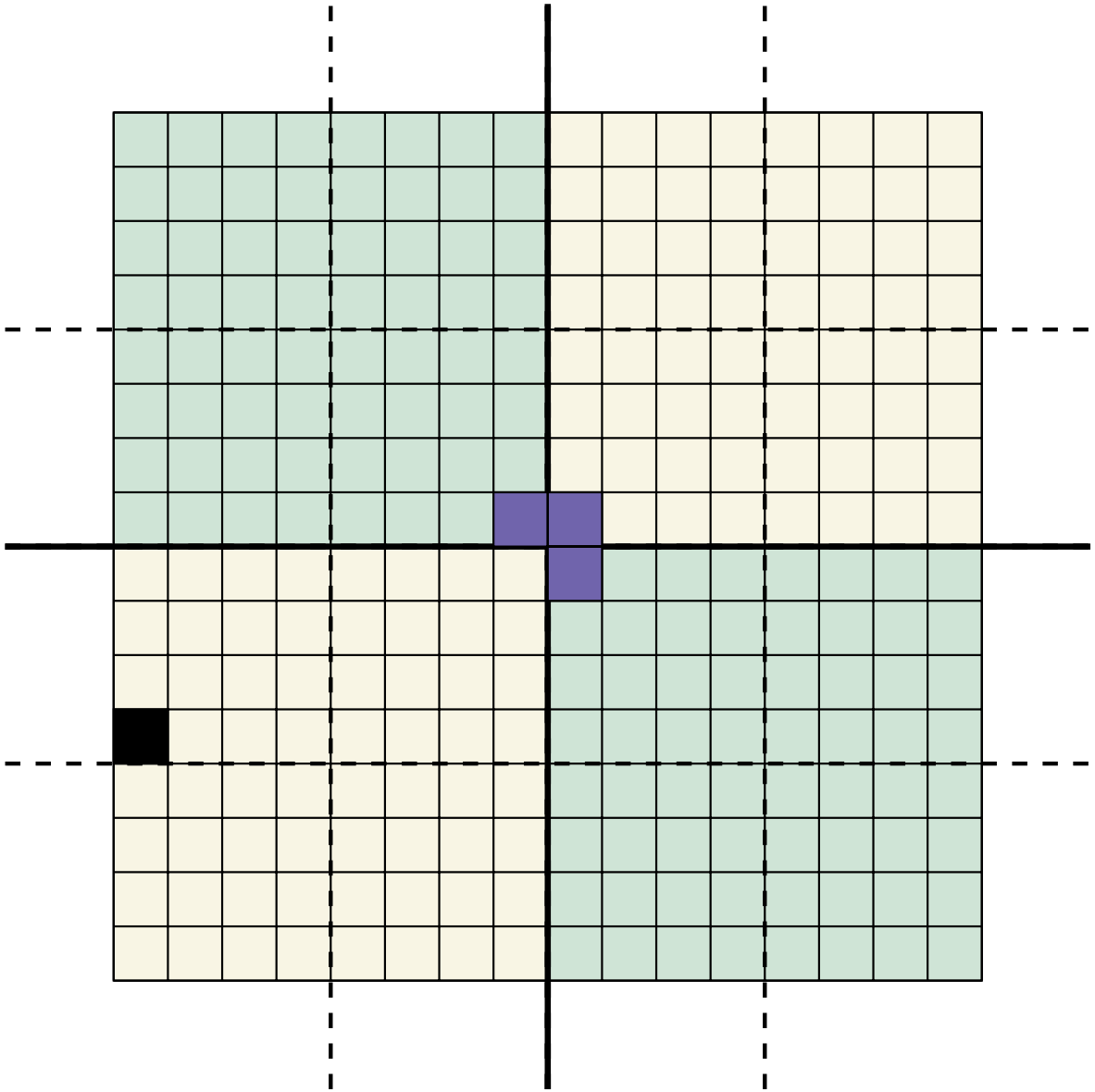


Triominoes tiling



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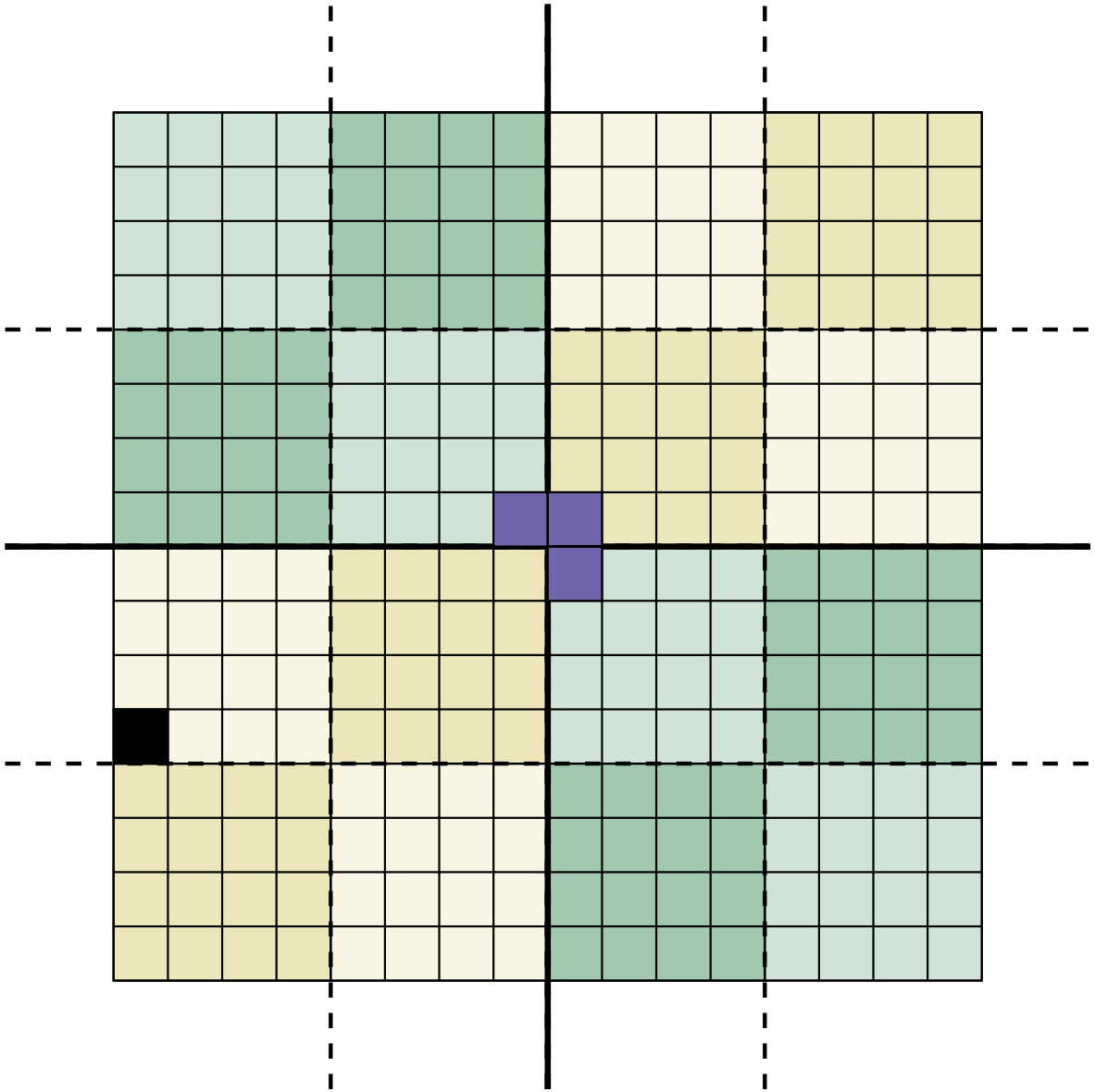

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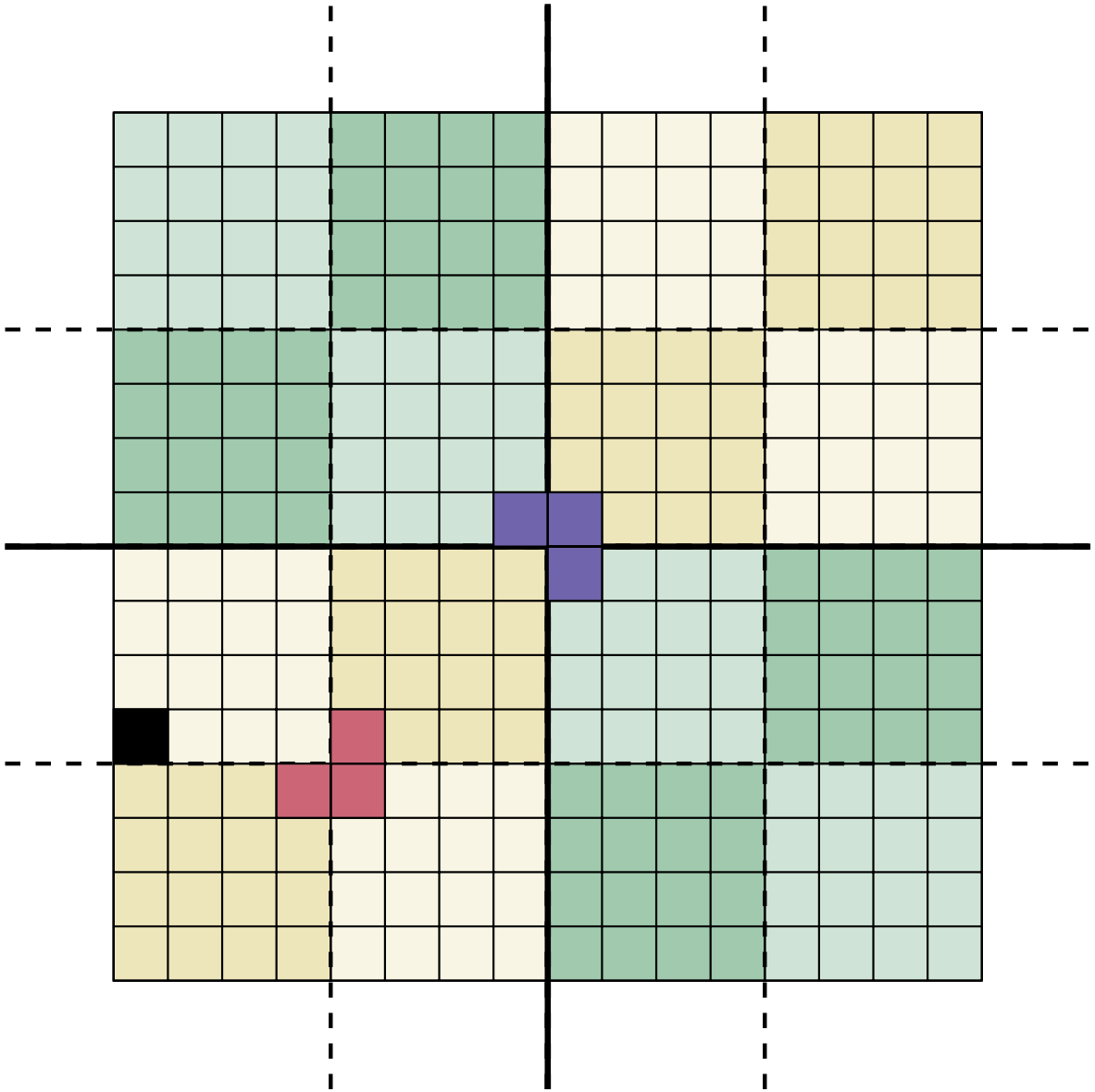

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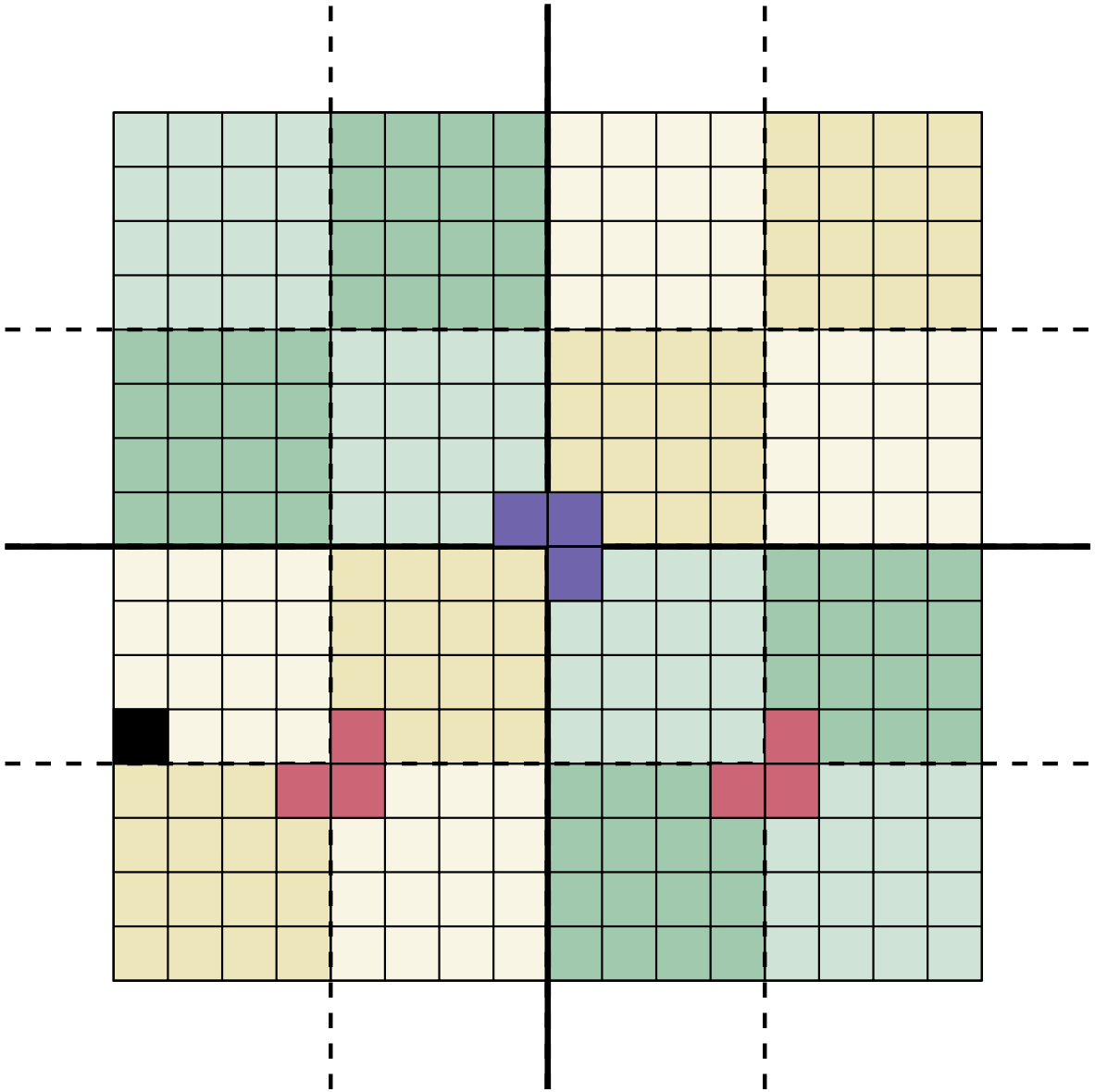

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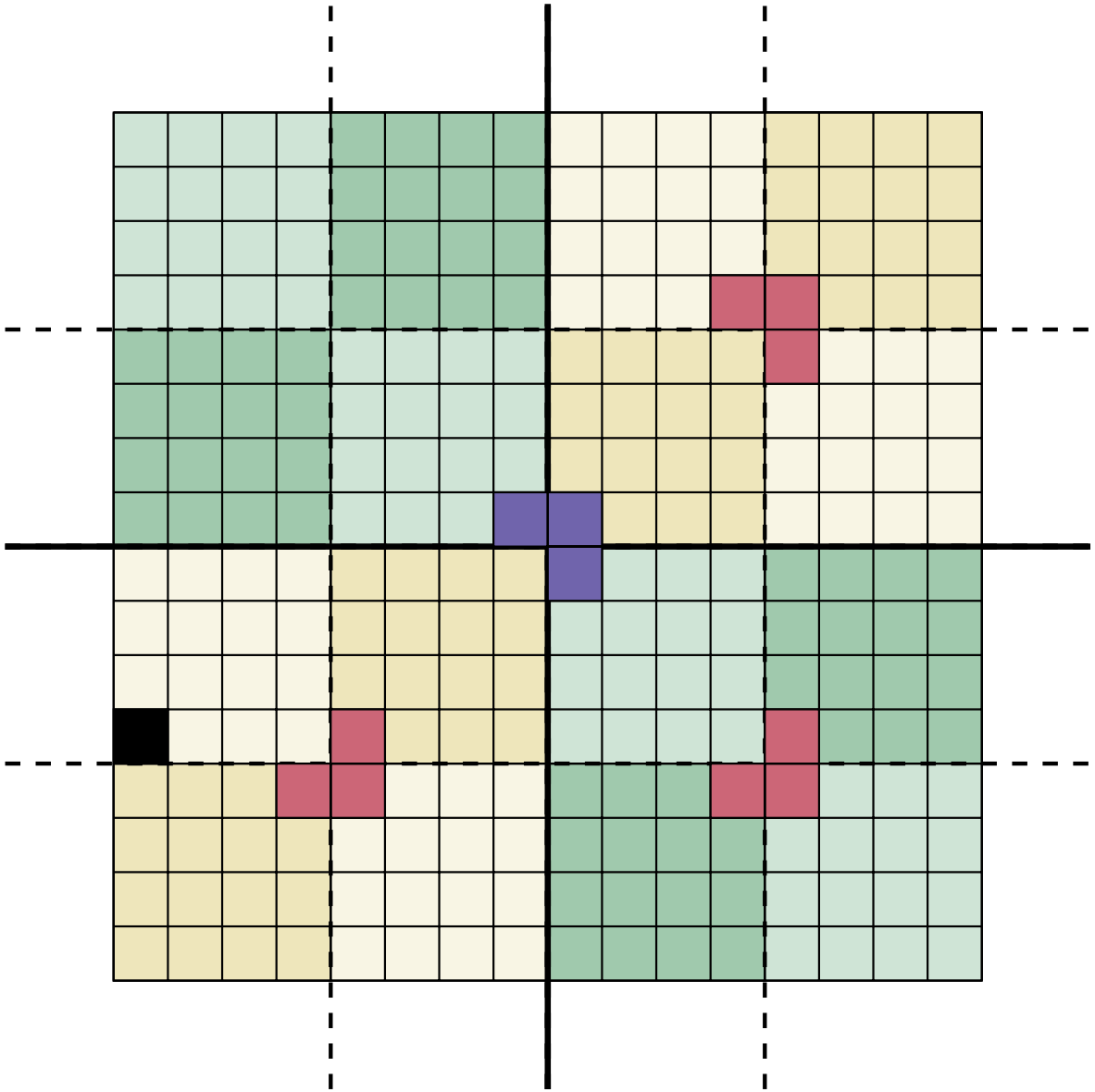

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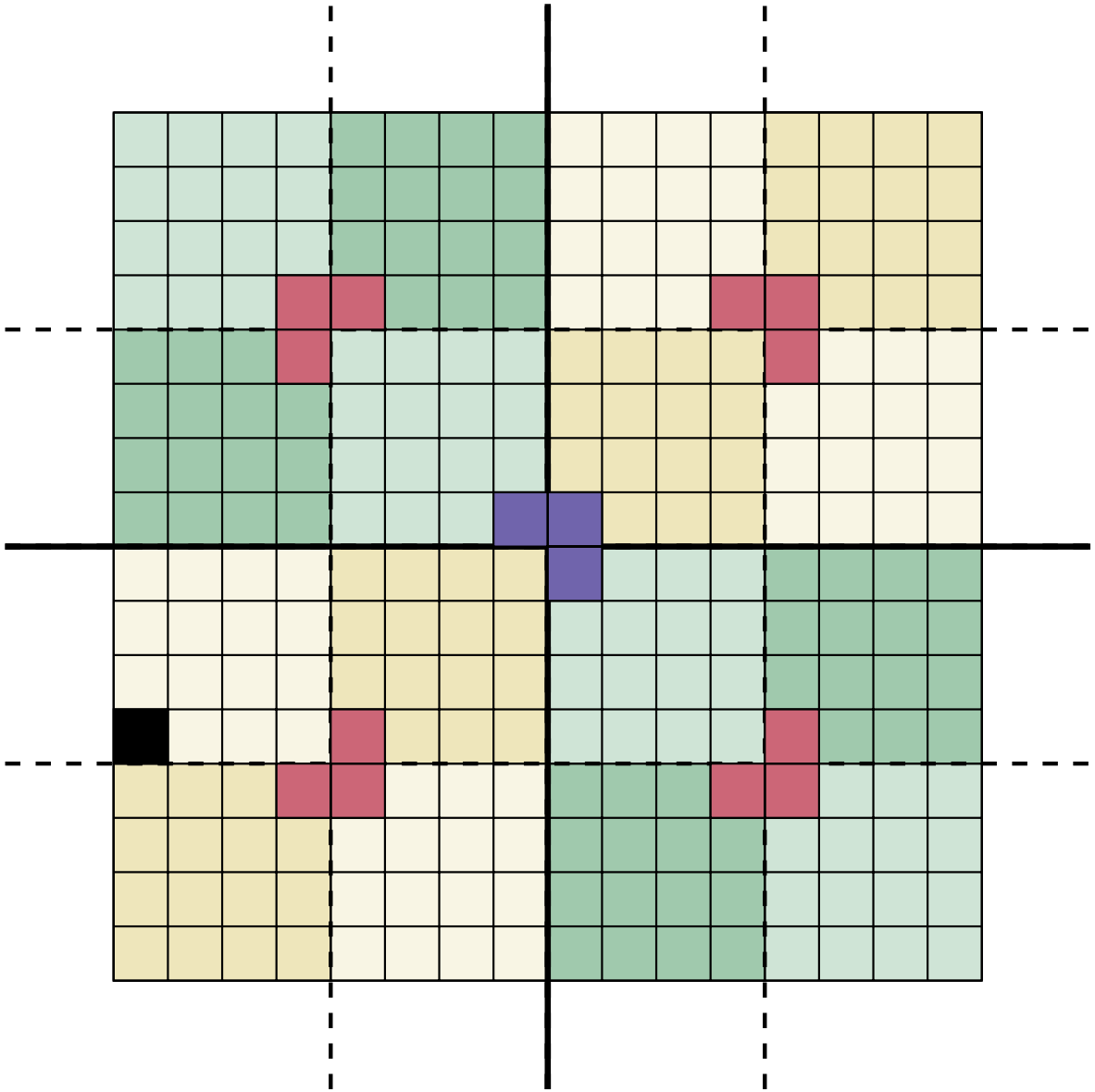

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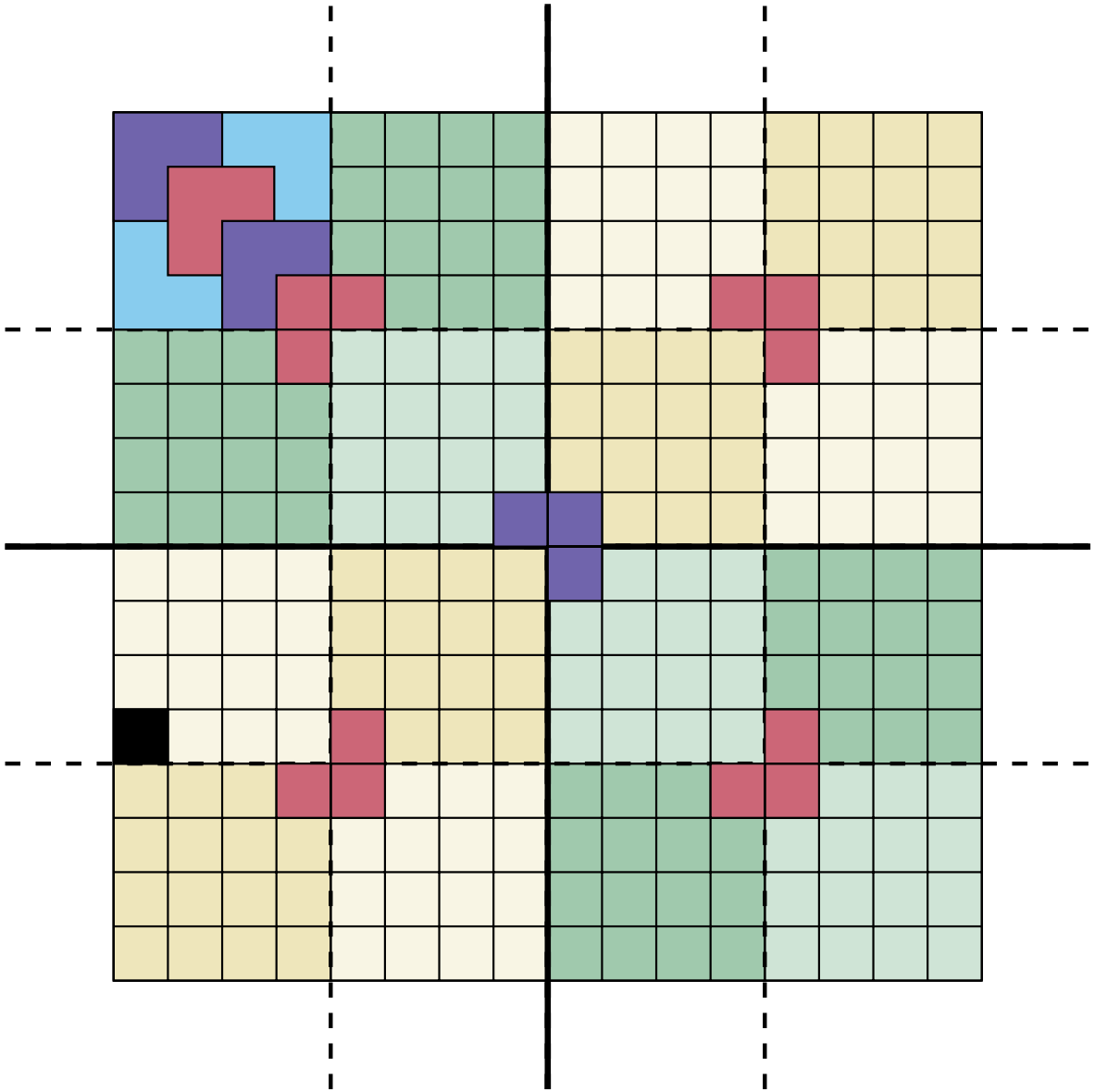

Triomino Corollary

10. Mathematical Induction
10.1 Strong Induction
10.2 Smallest Counterexample

Corollary: For any $n \in \mathbb{N}$, $3 | 2^{2n} - 1$.

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Proof: $2^{2n} - 1$ is the number of squares on a chessboard of dimension $2^n \times 2^n$ that's missing one piece. Since these squares can be covered completely by triomino pieces that have three squares each, $3 | 2^{2n} - 1$.

Prime Divisors

10. Mathematical Induction
10.1 Strong Induction
10.2 Smallest Counterexample

Proposition: Let p be prime. For any integers $a_1, \ldots, a_n$, if p divides the product $a_1 \cdots a_n$, then p divides some a_i, $1 \leq i \leq n$.

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Lemma: If a and b are integers with $\gcd(a, b) = 1$, then there exist $k, \ell \in \mathbb{Z}$ such that $ak + b\ell = 1$.

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Lemma: If a and b are integers with $\gcd(a, b) = 1$, then there exist $k, \ell \in \mathbb{Z}$ such that $ak + b\ell = 1$.

Proof: We proceed by induction on the number of integers. For the base case, suppose $n = 2$, and $p|a_1a_2$. If $p|a_1$, we're done. Suppose $p \nmid a_1$. Since p is prime, $\gcd(p, a_1) = 1$, so there exist $k, \ell \in \mathbb{Z}$ such that $kp + a_1\ell = 1$. Multiplying by a_2, we see

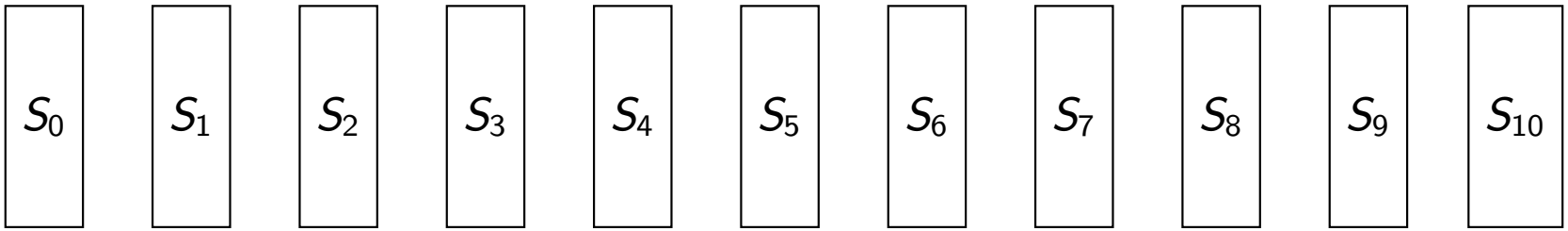
$$a_2 = kpa_2 + a_1a_2\ell.$$

Since $p|a_1a_2$, we see p divides the right hand side of the equation, so $p|a_2$. This finishes the base case.

Suppose now that the corollary holds whenever $n \leq m$ for some $m \in \mathbb{Z}$. Suppose $p|a_1 \cdots a_m a_{m+1}$. Let $A = a_1 \cdots a_m$. If $p|a_{m+1}$, we're done. If not, then since $p|A \cdot a_{m+1}$, by the base step, $p|A$. Then by the inductive hypothesis, p divides some a_i, $1 \leq i \leq m$. This finishes the proof.

Strong Induction

10. Mathematical Induction
10.1 Strong Induction
10.2 Smallest Counterexample



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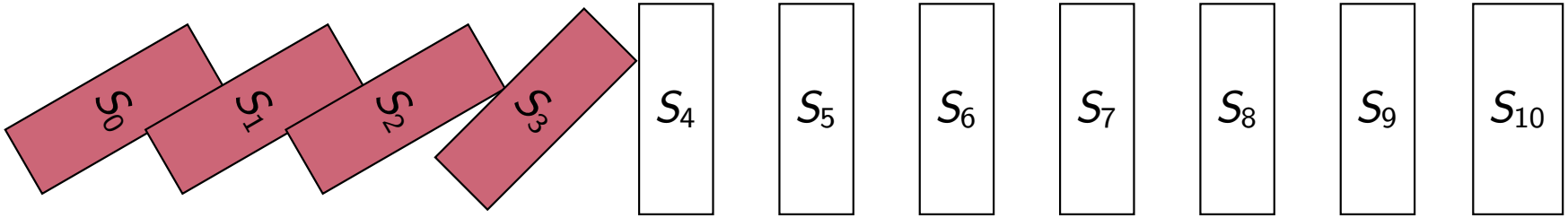

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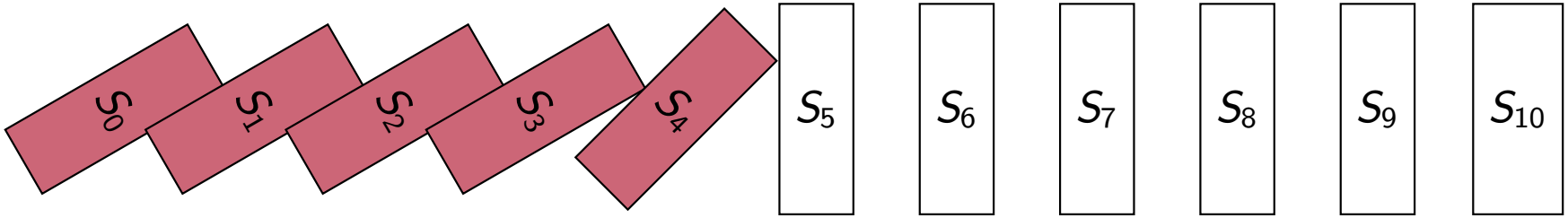

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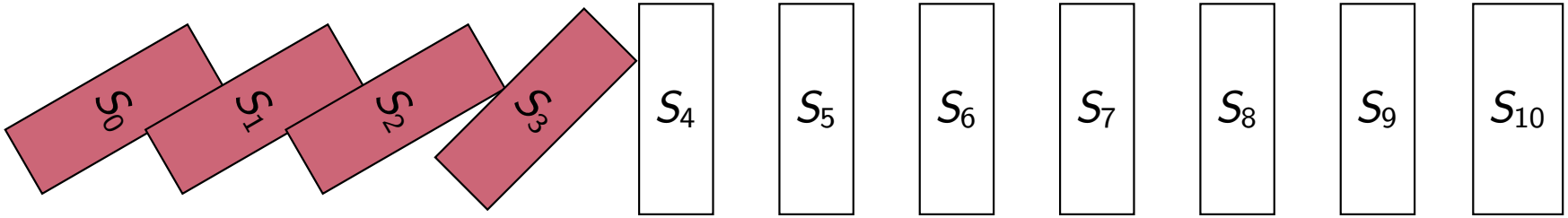

Outline for proof by strong induction:

- Suppose S_n is true for every $n \leq k$. Then S_{k+1} is true as well.
  This part is the inductive step;
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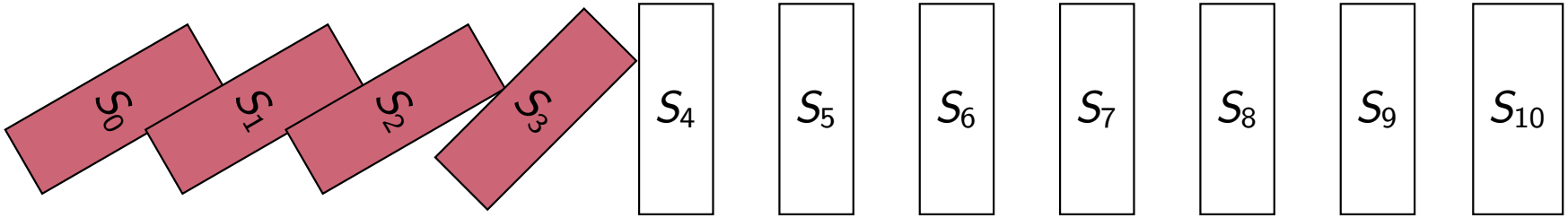
Strong Induction



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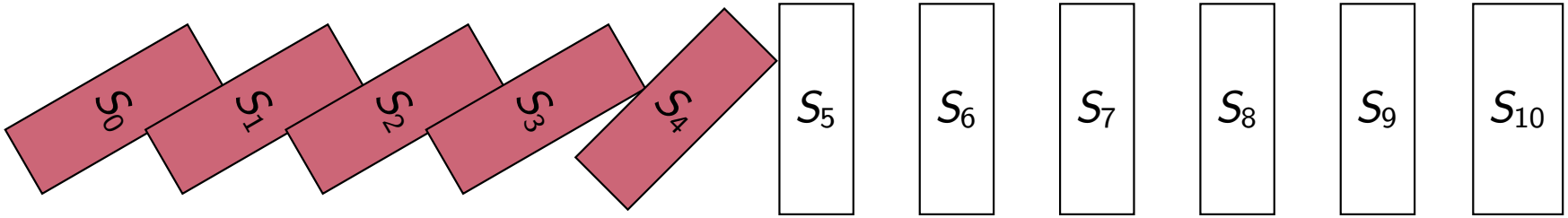

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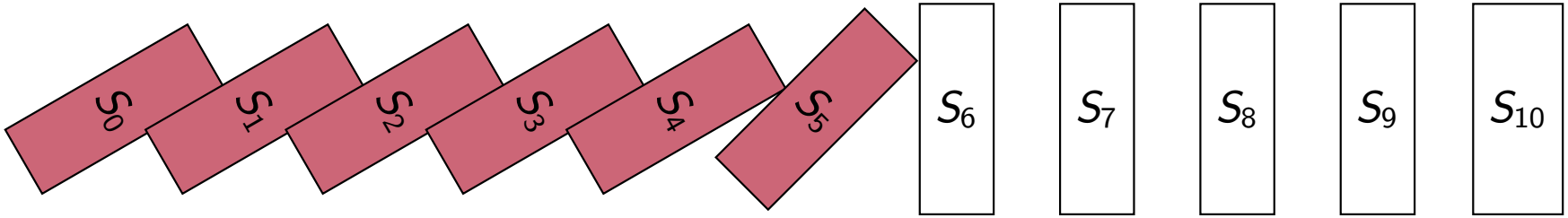

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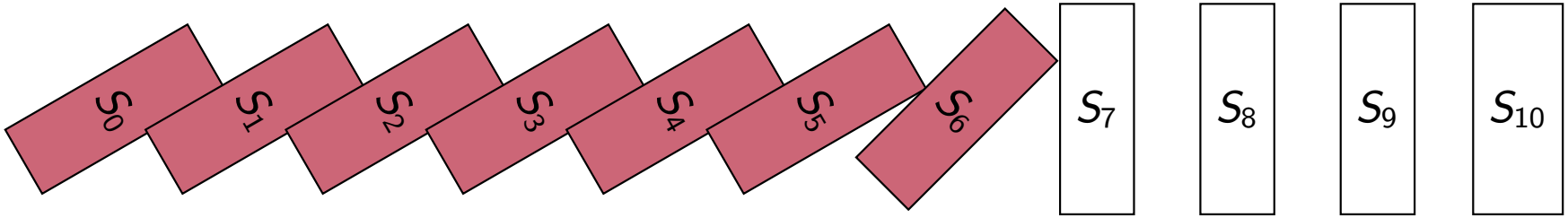

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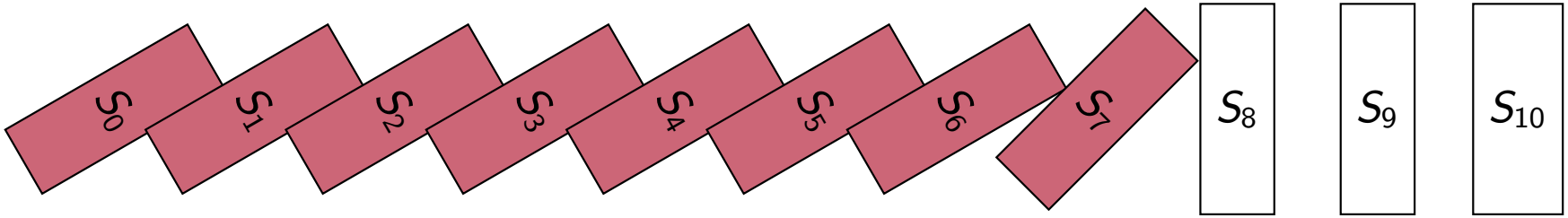

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Strong Induction

10. Mathematical Induction
10.1 Strong Induction
10.2 Smallest Counterexample

Proposition: Let $\{a_n\}$ be the sequence of numbers defined as follows:

- $a_1 = 2$
- $a_2 = 4$
- $a_3 = 8$
- $a_n = a_{n-1} + a_{n-2} + 2a_{n-3}$ for every natural $n \geq 4$.

Then $a_n = 2^n$ for every $n \in \mathbb{N}$.

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Then $a_n = 2^n$ for every $n \in \mathbb{N}$.

Proof:

- Suppose $a_n = 2^n$ for every natural $n \leq k$, and $k \geq 3$. Then $k + 1 \geq 4$, so:

$$\begin{aligned} a_{k+1} &= a_k + a_{k-1} + 2a_{k-2} \\ &= 2^k + 2^{k-1} + 2 \cdot 2^{k-2} = 2^k + 2 \cdot 2^{k-1} \\ &= 2 \cdot 2^k = 2^{k+1} \end{aligned}$$

  Then $a_{k+1} = 2^{k+1}$ for every natural $n \leq k + 1$.
- Note $a_1 = 2^1$, $a_2 = 2^2$, and $a_3 = 2^3$.
- By strong induction, we conclude $a_n = 2^n$ for every $n \in \mathbb{N}$.

Strong Induction: Nim

10. Mathematical Induction
10.1 Strong Induction
10.2 Smallest Counterexample

Game of Nim

There are two piles of matches, and two players. A turn consists of a player may take as many matches as they like (but at least one) from one of the piles. The player who takes the last match wins.

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Proposition: In a game of Nim where both piles of matches are initially the same size, the second player can always win if they take an equal number of matches as their opponent did in their last term, but from the other pile.

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Proposition: In a game of Nim where both piles of matches are initially the same size, the second player can always win if they take an equal number of matches as their opponent did in their last term, but from the other pile.

Proof: (by induction on the initial size of the piles)

- Suppose we know Player 2 can win when the sizes of the piles are less than m for some $m \in \mathbb{N}$. (Inductive Hypothesis) Player 1 takes k matches from one pile, for some $k \leq m$. If $k = m$, then Player 2 finishes off the remaining pile and wins. If $k \neq m$, then after Player 2's move, each pile has size $m - k$, and $1 \leq m - k < m$, so by the IH, Player 2 can now win.
- When each pile has size 1, Player 2 wins on their first move.
- We conclude by induction that the proposition is true.

Strong Induction

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10.1 Strong Induction
10.2 Smallest Counterexample

Proposition: For any convex polygon with $n \geq 3$ edges, the sum of the interior angles is $(n-2)180$ degrees.

Lemma: If you draw a line between any two (non-consecutive) points on a convex polygon, you divide that convex polygon into two (smaller) convex polygons.

Lemma: The interior angles of any triangle sum to 180 degrees.

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Lemma: The interior angles of any triangle sum to 180 degrees.

- First, note that a triangle has angles summing to 180 degrees.
- Suppose the proposition holds whenever $n \leq k-1$. Then consider any polygon with k edges. Draw a point between two ponts that are separated by only one point, so the polygon is cut into a triangle and a convex polygon on $k-1$ points. Then the sum of the angles inside the triangle is 180, while the sum of the angles inside the other polygon is (by the inductive hypothesis) $180(k-3)$. Then the sum of the inside angles of the original polygon is $180 + 180(k-3) = 180(k-2)$, so the proposition also holds for $n = k$.
- By strong induction, we conclude the proposition holds for all natural $n \geq 3$.

Proof by Smallest Counterexample

10. Mathematical Induction
10.1 Strong Induction
10.2 Smallest Counterexample

Proposition: The statements $S_1, S_2, \ldots$ are all true.

Proof:

- S_1 is true

Proof by Smallest Counterexample

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Proposition: The statements $S_1, S_2, \ldots$ are all true.

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- $\vdots$
- Contradiction.

Proof by Smallest Counterexample: Fibonnaci Numbers

Fibonnaci Sequence

$F_0 = 1$ • $F_1 = 1$ • $F_2 = 2$ • For $n \geq 3$ in $\mathbb{N}$, $F_n = F_{n-1} + F_{n-2}$

Proof by Smallest Counterexample: Fibonnaci Numbers

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Let φ be the golden ratio, $\varphi = \dfrac{1+\sqrt{5}}{2}$.

Proposition: For every $n \in \mathbb{N}$, $F_n < \varphi^n$.

Proof by Smallest Counterexample: Fibonnaci Numbers

Let φ be the golden ratio, $\varphi = \dfrac{1+\sqrt{5}}{2}$.

Proposition: For every $n \in \mathbb{N}$, $F_n < \varphi^n$. *Proof:*

- Note that $\phi > \frac{3}{2}$, so $F_1 < \phi^1$ and $F_2 < \frac{9}{4} < \phi^2$. (Base cases.)
- Suppose F_n is the first Fibonnaci number such that $F_n \geq \varphi^n$. Then $n \geq 3$ (by the base cases) and so $F_{n-1} < \varphi^{n-1}$ and $F_{n-2} < \varphi^{n-2}$. Then:

$$\begin{aligned}
\varphi^n \leq F_n = F_{n-1} + F_{n-2} &< \varphi^{n-1} + \varphi^{n-2} \\
\varphi^2 &< \varphi + 1 \\
\left(\frac{1+\sqrt{5}}{2}\right)^2 &< \frac{1+\sqrt{5}}{2} + 1 \\
\frac{6+2\sqrt{5}}{4} &< \frac{3+\sqrt{5}}{2} \\
3+\sqrt{5} &< 3+\sqrt{5}
\end{aligned}$$

  This is a contradiction.

- We conclude that there does not exist any $n \in \mathbb{N}$ such that $F_n \geq \varphi$. Therefore, the proposition is true.

Proof by Smallest Counterexample: $4|5^n - 1$

10. Mathematical Induction
10.1 Strong Induction
10.2 Smallest Counterexample

Proposition: For every $n \in \mathbb{N}$, $4|5^n - 1$.

Proof by Smallest Counterexample: $4|5^n - 1$

Proposition: For every $n \in \mathbb{N}$, $4|5^n - 1$.

Proof:

- Note that when $n = 1$, $5|5^1 - 1$.
- Suppose n is the smallest natural number so that $4 \nmid 5^n - 1$. Then $n \geq 2$ (by the base case), so $n - 1 \in \mathbb{N}$ (that is–it isn't zero), so by hypothesis, $4|5^{n-1} - 1$. That is, there exists some $x \in \mathbb{N}$ such that $4x = 5^{n-1} - 1$. Now:

$$\begin{aligned} 5^n - 1 &= 5 \cdot 5^{n-1} - 1 = 5\,(4x + 1) - 1 \\ &= 5(4x) + 5 - 1 = 4(5x) + 4 = 4(5x + 1) \end{aligned}$$

  This contradicts the assumption that $4 \nmid 5^n - 1$.
- We conclude that there is no natural n such that $4 \nmid 5^n - 1$, so the proposition is true.

Proof by Smallest Counterexample: Fundamental Theorem of Arithmetic

Fundamental Theorem of Arithmetic

Every $n \in \mathbb{N}$ greater than 1 has a unique prime factorization. That is, if $n = p_1 \cdot p_2 \cdot p_3 \cdots p_k$ and $n = a_1 \cdot a_2 \cdot a_3 \cdots a_\ell$ are two prime factorizations of n, then $k = \ell$, and the primes p_i and a_i are the same (but they might be in a different order).

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This proof involves both existence and uniqueness.

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This proof involves both existence and uniqueness.

Proof: First, we will use strong induction to show that every integer $n > 1$ has a prime factorization. For the base case, suppose $n = 2$. Then $2 = 2$ is the prime factorization of n. For the inductive step, suppose there exists an integer $k \geq 2$ such that that every integer n in $[2, k]$ has a prime factorization. We consider the integer $k + 1$. If $k + 1$ is prime, then it is its own prime factorization. If it is not prime, then $k + 1 = ab$ for some positive integers a and b, both of which are greater than 1 and less than $k + 1$. By the inductive hypothesis, $a = p_1 \cdots p_k$ and $b = a_1 \cdots a_\ell$ for primes p_i, a_i. Then $k + 1 = p_1 \cdots p_k \cdot a_1 \cdots a_\ell$. So, $k + 1$ has a prime factorization. We conclude by induction that every integer greater than 1 has a prime factorization.

Fundamental Theorem of Arithmetic, Cont'd

Now we will prove uniqueness, using proof by smallest counterexample. For the base case, note that $n = 2$ has only one prime factorization, since the only prime divisor of 2 is itself. Suppose for the sake of contradiction that there is some integer $n > 2$ with different prime factorizations. Let n be, in particular, the smallest such integer, and label the factorizations $n = p_1 \cdots p_k$ and $a_1 \cdots a_\ell$. Then $p_1 | n$, so $p_1 | a_1 \cdots a_\ell$. Then by a previous theorem, $p_1 | a_i$ for some $1 \leq i \leq \ell$. Then, since a_i is prime, $a_i = p_1$. But now the integer $\frac{n}{p_1}$ has two different prime factorizations, $p_2 \cdots p_k$ and $a_1 \cdots a_{i-1} a_{i+1} \cdots a_\ell$. This contradicts that n was the smallest such integer, and finishes the proof.