Week 9: complex numbers; complex exponential and polar form

Course Notes: 5.1, 5.2, 5.3, 5.4

Goals:
Fluency with arithmetic on complex numbers
Using matrices with complex entries: finding determinants and inverses, solving systems, etc.
Visualizing complex numbers in coordinate systems

Complex Arithmetic

We use $i$ (as in "imaginary") to denote the number whose square is $-1$.

When we talk about “complex numbers,” we allow numbers to have real parts and imaginary parts:

$2 + 3i$, $-1$, and $2i$

Addition happens component-wise, just like with vectors or polynomials.
Complex Arithmetic

Multiplication is similar to polynomials.

\[ \begin{align*}
\text{I: } & 0 \\
A: & (-4 + 3i) + (1 - i) \\
\text{II: } & -1 \\
B: & i(2 + 3i) \\
\text{III: } & -2 \\
C: & (i + 1)(i - 1) \\
\text{IV: } & 2i + 12 \\
D: & (2i + 3)(i + 4) \\
\text{V: } & -3 + 2i \\
E: & 10 + 11i \\
\text{VI: } & 3 + 2i \\
\text{VII: } & 10 + 11i
\end{align*} \]

Notes

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \bar{x + yi} = x - yi \]
5.1: Complex Arithmetic

$$\frac{z}{w} = \frac{z \overline{w}}{|w|^2}$$

Compute:

5.2: Complex Matrices and Linear Systems

5.3: Complex Exponential

5.4: Polar Representation

Polynomial Factorizations

**Theorem**

Every polynomial can be factored completely over the complex numbers.

Example: $x^2 + 1 = (x - i)(x + i)$

Calculating Determinants

We calculate the determinant of a matrix with complex entries in the same way we calculate the determinant of a matrix with real entries.

$$\det \begin{bmatrix} 1 + i & 1 - i \\ 2 & i \end{bmatrix} =$$

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Gaussian Elimination

Give a parametric equation for all solutions to the homogeneous system:
\[ \begin{align*}
ix_1 + x_2 + 2x_3 &= 0 \\
ix_2 + 3x_3 &= 0 \\
2ix_3 + (2-i)x_2 + x_3 &= 0
\end{align*} \]

Solve the following system of equations:
\[ \begin{align*}
ix_1 + 2x_2 &= 9 \\
3x_1 + (1+i)x_2 &= 5 + 8i
\end{align*} \]

Find the inverse of the matrix
\[
\begin{bmatrix}
i & 1 \\2 & 3i
\end{bmatrix}
\]

Exponentials

What to do when \(i\) is the power of a function?
Maclaurin (Taylor) Series:
\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots
\]
\[
sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\]
\[
cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\]
\[
e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots
\]
\[
= 1 + ix - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots
\]
\[
= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right) + i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right)
\]
\[
= \cos x + i \sin x
\]

Does that even make sense?
\[
e^{ix} = \cos x + i \sin x
\]
\[
\frac{d}{dx}(e^{ix}) = ie^{ix},
\]
\[
e^{x+y} = e^x e^y;
\]
Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

Simplify:

\[ e^{\frac{\pi}{2}i} \]
\[ e^{2 + i} \]
\[ \sqrt{2}e^{\frac{\pi}{4}i} \]
\[ 2i \]
\[ e^{\pi i} + 1 \]

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Complex exponentiation: \( e^{ix} = \cos x + i \sin x \)

Let \( x \) be a real number.
True or False?

1. \( e^x = \cos x \)
2. \( e^{ix} = e^{i(x+2\pi)} \)
3. \( e^{ix} = -e^{i(x+\pi)} \)
4. \( e^{ix} + e^{-ix} \) is a real number

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Coordinates Revisited

Complex number: \( r(\cos \theta + i \sin \theta) = re^{i\theta} = re^{i(\theta + 2\pi)} \)
### Coordinates Revisited

Geometric interpretation of multiplication of two complex numbers: add the angles, multiply the lengths (moduli).

$$re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta + \phi)}$$

### Roots of Unity

Roots:

\[
e^{i\frac{2\pi}{3}}
\]

\[
e^{i\frac{4\pi}{3}}
\]

\[
(re^{i\theta})^3 = 1
\]

Notes

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### 5.1: Complex Arithmetic

- Find all complex numbers $z$ such that $z^3 = 8$.
- Find all complex numbers $z$ such that $z^3 = 27e^{i\frac{2\pi}{3}}$.
- Find all complex numbers $z$ such that $z^4 = 81e^{2i}$. 

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