Outline

Week 9: complex numbers; complex exponential and polar form

Course Notes: 5.1, 5.2, 5.3, 5.4

Goals:
Fluency with arithmetic on complex numbers
Using matrices with complex entries: finding determinants and inverses, solving systems, etc.
Visualizing complex numbers in coordinate systems
**Complex Arithmetic**

We use $i$ (as in "imaginary") to denote the number whose square is $-1$. 

\[ i^2 = -1 \] 
\[ i^3 = -i \] 
\[ i^4 = 1 \]
We use $i$ (as in "imaginary") to denote the number whose square is $-1$.

$i^2 = -1$
Complex Arithmetic

We use $i$ (as in "imaginary") to denote the number whose square is $-1$.

$$i^2 = -1 \quad (-i)^2 =$$
We use $i$ (as in "imaginary") to denote the number whose square is $-1$. 

\[ i^2 = -1 \quad (\bar{i})^2 = -1 \]
We use \( i \) (as in "imaginary") to denote the number whose square is \(-1\).

\[
\begin{align*}
    i^2 &= -1 \\
    (-i)^2 &= -1 \\
    i^3 &= \text{(to be determined)}
\end{align*}
\]
Complex Arithmetic

We use $i$ (as in "imaginary") to denote the number whose square is $-1$.

\[
i^2 = -1 \quad (-i)^2 = -1 \quad i^3 = -i
\]
We use $i$ (as in "imaginary") to denote the number whose square is $-1$.

\[ i^2 = -1 \quad (-i)^2 = -1 \quad i^3 = -i \quad i^4 = \]
Complex Arithmetic

We use $i$ (as in "imaginary") to denote the number whose square is $-1$.

\[
i^2 = -1 \quad (-i)^2 = -1 \quad i^3 = -i \quad i^4 = 1
\]
We use $i$ (as in "imaginary") to denote the number whose square is $-1$. 

\[ i^2 = -1 \quad (\text{and} \quad (-i)^2 = -1 \quad i^3 = -i \quad i^4 = 1 \]

When we talk about "complex numbers," we allow numbers to have real parts and imaginary parts:

\[ 2 + 3i \quad -1 \quad 2i \]
Complex Arithmetic

\[ 2 + 3i \quad -1 \quad 2i \]

imaginary

real
Complex Arithmetic

\[2 + 3i \quad -1 \quad 2i\]

imaginary

2 + 3i

real

3

2
Complex Arithmetic

2 + 3i

−1

2i

2 + 3i
Complex Arithmetic

\[ 2 + 3i - 1 = 2i \]

Diagram:

- Real axis
- Imaginary axis
- Origin
- Vector from origin to point \(2 + 3i\)
- Vector from origin to point \(-1\)
- Vector from origin to point \(2i\)
- Vector from origin to point \(2\)
- Vector from origin to point \(3\)
Complex Arithmetic

Addition happens component-wise, just like with vectors or polynomials.
Complex Arithmetic

Addition happens component-wise, just like with vectors or polynomials.

\[(2 + 3i) + (3 - 4i) =\]
Complex Arithmetic

Addition happens component-wise, just like with vectors or polynomials.

\[(2 + 3i) + (3 - 4i) = 5 - i\]
Complex Arithmetic

Multiplication is similar to polynomials.
Complex Arithmetic

Multiplication is similar to polynomials.

\[(2 + 3i)(3 - 4i) =\]
Complex Arithmetic

Multiplication is similar to polynomials.

\[(2 + 3i)(3 - 4i) = 2 \cdot 3 + 3i \cdot 3 + (2)(-4i) + (3i)(-4i)\]
Complex Arithmetic

Multiplication is similar to polynomials.

\[(2 + 3i)(3 - 4i) = 2 \cdot 3 + 3i \cdot 3 + (2)(-4i) + (3i)(-4i)\]
\[= 6 + 9i - 8i + 12\]
Multiplication is similar to polynomials.

\[(2 + 3i)(3 - 4i) = 2 \cdot 3 + 3i \cdot 3 + (2)(-4i) + (3i)(-4i)\]
\[= 6 + 9i - 8i + 12 = 18 + i\]
Complex Arithmetic

Multiplication is similar to polynomials.

\[(2 + 3i)(3 - 4i) = 2 \cdot 3 + 3i \cdot 3 + (2)(-4i) + (3i)(-4i)\]

\[= 6 + 9i - 8i + 12 = 18 + i\]

A: \((-4 + 3i) + (1 - i)\)

B: \(i(2 + 3i)\)

C: \((i + 1)(i - 1)\)

D: \((2i + 3)(i + 4)\)

I: 0

II: -1

III: -2

IV: 2i + 12

V: -3 + 2i

VI: 3 + 2i

VII: 10 + 11i
Complex Arithmetic

Multiplication is similar to polynomials.

\[(2 + 3i)(3 - 4i) = 2 \cdot 3 + 3i \cdot 3 + (2)(-4i) + (3i)(-4i)\]

\[= 6 + 9i - 8i + 12 = 18 + i\]
Complex Arithmetic

Modulus

The **modulus** of \((x + yi)\) is:

\[ |x + yi| = \sqrt{x^2 + y^2} \]

like the norm of a vector.
Complex Arithmetic

Modulus

The *modulus* of \( (x + yi) \) is:

\[
|x + yi| = \sqrt{x^2 + y^2}
\]

like the norm of a vector.

Complex Conjugate

The *complex conjugate* of \( (x + yi) \) is:

\[
\overline{x + yi} = x - yi
\]

the reflection of the vector over the real (x) axis.
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \]

\[ x + yi = x - yi \]
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \bar{x + yi} = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

- \( z - \bar{z} \)
- \( z + \bar{z} \)
- \( z\bar{z} - |z|^2 \)
- \( \bar{z}w - (\bar{z})(\bar{w}) \)
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \quad \quad \quad x + yi = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

- \( z - \bar{z} = 2yi \)  
  \( y \) is called the imaginary part of \( z \)
- \( z + \bar{z} \)
- \( z\bar{z} - |z|^2 \)
- \( \bar{z} w - (\bar{z})(\bar{w}) \)
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \]

\[ x + yi = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

- \( z - \bar{z} = 2yi \) \( y \) is called the imaginary part of \( z \)
- \( z + \bar{z} = 2x \) \( x \) is called the real part of \( z \)
- \( z\bar{z} - |z|^2 \)
- \( \bar{z}w - (\bar{z})(\bar{w}) \)
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \quad x + yi = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

- \( z - \bar{z} = 2yi \) \( y \) is called the imaginary part of \( z \)
- \( z + \bar{z} = 2x \) \( x \) is called the real part of \( z \)
- \( z\bar{z} - |z|^2 = 0 \) So, \( z\bar{z} = |z|^2 \)
- \( \bar{z}w - (\bar{z})(w) \)
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \text{and} \quad x + yi = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

- \( z - \overline{z} = 2yi \)  
  \( y \) is called the imaginary part of \( z \)

- \( z + \overline{z} = 2x \)  
  \( x \) is called the real part of \( z \)

- \( z\overline{z} - |z|^2 = 0 \)  
  So, \( z\overline{z} = |z|^2 \)

- \( \overline{zw} - (\overline{z})(\overline{w}) = 0 \)  
  So, \( \overline{zw} = \overline{z} \overline{w} \)
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \text{and} \quad x + yi = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

- \( z - \bar{z} = 2yi \) \( y \) is called the imaginary part of \( z \)
- \( z + \bar{z} = 2x \) \( x \) is called the real part of \( z \)
- \( z\bar{z} - |z|^2 = 0 \) So, \( z\bar{z} = |z|^2 \)
- \( \overline{zw} - (\bar{z})(\bar{w}) = 0 \) So, \( \overline{zw} = \bar{z} \bar{w} \)

Division

\[
\frac{z}{w} = \]

Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \quad \quad \quad x + yi = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

- \( z - \bar{z} = 2yi \) \( y \) is called the imaginary part of \( z \)
- \( z + \bar{z} = 2x \) \( x \) is called the real part of \( z \)
- \( z\bar{z} - |z|^2 = 0 \) So, \( z\bar{z} = |z|^2 \)
- \( \bar{z}\bar{w} - (\bar{z})(\bar{w}) = 0 \) So, \( \bar{z}\bar{w} = \bar{z} \bar{w} \)

Division

\[ \frac{z}{w} = \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} \]
5.1: Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \quad x + yi = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

- \( z - \bar{z} = 2yi \) \quad \text{\( y \) is called the imaginary part of \( z \)}
- \( z + \bar{z} = 2x \) \quad \text{\( x \) is called the real part of \( z \)}
- \( z\bar{z} - |z|^2 = 0 \) \quad \text{So, } z\bar{z} = |z|^2
- \( \bar{z}w - (\bar{z})(\bar{w}) = 0 \) \quad \text{So, } \bar{z}w = z\bar{w}

### Division

\[
\frac{z}{w} = \frac{z}{w} \cdot \frac{\bar{w}}{\bar{w}} = \frac{zw}{|w|^2}
\]
Complex Arithmetic

\[
\frac{z}{w} = \frac{zw}{|w|^2}
\]
Complex Arithmetic

\[ \frac{z}{w} = \frac{zw}{|w|^2} \]

Compute:

- \[ \frac{2+3i}{3+4i} \]
- \[ \frac{1+3i}{1-3i} \]
- \[ \frac{2}{1+i} \]
- \[ \frac{5}{i} \]
Complex Arithmetic

\[ \frac{z}{w} = \frac{zw}{|w|^2} \]

Compute:

- \[ \frac{2+3i}{3+4i} = \frac{18}{25} + \frac{1}{25}i \]
- \[ \frac{1+3i}{1-3i} \]
- \[ \frac{2}{1+i} \]
- \[ \frac{5}{i} \]
Complex Arithmetic

\[
\frac{z}{w} = \frac{zw}{|w|^2}
\]

Compute:

- \( \frac{2+3i}{3+4i} = \frac{18}{25} + \frac{1}{25}i \)
- \( \frac{1+3i}{1-3i} = \frac{-4}{5} + \frac{3}{5}i \)
- \( \frac{2}{1+i} \)
- \( \frac{5}{i} \)
Complex Arithmetic

\[
\frac{z}{w} = \frac{zw}{|w|^2}
\]

Compute:

\[
\begin{align*}
\frac{2+3i}{3+4i} &= \frac{18}{25} + \frac{1}{25}i \\
\frac{1+3i}{1-3i} &= \frac{-4}{5} + \frac{3}{5}i \\
\frac{2}{1+i} &= 1 - i \\
\frac{5}{i} &= \frac{5i}{i^2} = 5i
\end{align*}
\]
### Complex Arithmetic

\[
\frac{z}{w} = \frac{zw}{|w|^2}
\]

Compute:

1. \( \frac{2+3i}{3+4i} = \frac{18}{25} + \frac{1}{25}i \)
2. \( \frac{1+3i}{1-3i} = -\frac{4}{5} + \frac{3}{5}i \)
3. \( \frac{2}{1+i} = 1 - i \)
4. \( \frac{5}{i} = -5i \)
Polynomial Factorizations

Theorem

Every polynomial can be factored completely over the complex numbers.
Polynomial Factorizations

Theorem

Every polynomial can be factored completely over the complex numbers.

Example: \(x^2 + 1 = (x - i)(x + i)\)
Polynomial Factorizations

**Theorem**

Every polynomial can be factored completely over the complex numbers.

Example: \( x^2 + 1 = (x - i)(x + i) \)

Example: \( x^2 + 2x + 10 = \)
Polynomial Factorizations

Theorem

Every polynomial can be factored completely over the complex numbers.

Example: \( x^2 + 1 = (x - i)(x + i) \)

Example: \( x^2 + 2x + 10 = (x + 1 + 3i)(x + 1 - 3i) \)
Polynomial Factorizations

Theorem

Every polynomial can be factored completely over the complex numbers.

Example: \( x^2 + 1 = (x - i)(x + i) \)

Example: \( x^2 + 2x + 10 = (x + 1 + 3i)(x + 1 - 3i) \)

Example: \( x^2 + 4x + 5 = \)
Theorem

Every polynomial can be factored completely over the complex numbers.

Example: $x^2 + 1 = (x - i)(x + i)$

Example: $x^2 + 2x + 10 = (x + 1 + 3i)(x + 1 - 3i)$

Example: $x^2 + 4x + 5 = (x + 2 + i)(x + 2 - i)$
Calculating Determinants

We calculate the determinant of a matrix with complex entries in the same way we calculate the determinant of a matrix with real entries.

\[
\begin{vmatrix}
1 + i & 1 - i \\
i & 2
\end{vmatrix} = (1 + i)(2) - (1 - i)(i)
\]

\[
= -3 + 3i
\]
Calculating Determinants

We calculate the determinant of a matrix with complex entries in the same way we calculate the determinant of a matrix with real entries.

\[
\det \begin{bmatrix}
1 + i & 1 - i \\
2 & i
\end{bmatrix}
\]
Calculating Determinants

We calculate the determinant of a matrix with complex entries in the same way we calculate the determinant of a matrix with real entries.

\[
\det \begin{bmatrix} 1 + i & 1 - i \\ 2 & i \end{bmatrix} = (1 + i)(i) - (1 - i)(2) =
\]
Calculating Determinants

We calculate the determinant of a matrix with complex entries in the same way we calculate the determinant of a matrix with real entries.

\[
\det \begin{bmatrix} 1 + i & 1 - i \\ 2 & i \end{bmatrix} = (1 + i)(i) - (1 - i)(2) = -3 + 3i
\]
Gaussian Elimination

Give a parametric equation for all solutions to the homogeneous system:

\[ \begin{align*}
x_1 &+ x_2 + 2x_3 = 0 \\
ix_2 &+ 3x_3 = 0 \\
2ix_1 &+ (2 - i)x_2 + x_3 = 0
\end{align*} \]
Gaussian Elimination

Give a parametric equation for all solutions to the homogeneous system:

\[ \begin{align*}
ix_1 + x_2 + 2x_3 &= 0 \\
i x_2 + 3x_3 &= 0 \\
2ix_1 + (2 - i)x_2 + x_3 &= 0
\end{align*} \]

Solve the following system of equations:

\[ \begin{align*}
ix_1 + 2x_2 &= 9 \\
3x_1 + (1 + i)x_2 &= 5 + 8i
\end{align*} \]
Gaussian Elimination

Give a parametric equation for all solutions to the homogeneous system:

\[ \begin{aligned}
ix_1 &+ x_2 + 2x_3 &= 0 \\
ix_2 &+ 3x_3 &= 0 \\
2ix_1 &+ (2 - i)x_2 + x_3 &= 0
\end{aligned} \]

Solve the following system of equations:

\[ \begin{aligned}
ix_1 &+ 2x_2 &= 9 \\
3x_1 &+ (1 + i)x_2 &= 5 + 8i
\end{aligned} \]

Find the inverse of the matrix
\[
\begin{bmatrix}
i & 1 \\
2 & 3i
\end{bmatrix}
\]
Gaussian Elimination

Give a parametric equation for all solutions to the homogeneous system:

\[ ix_1 + x_2 + 2x_3 = 0 \]
\[ ix_2 + 3x_3 = 0 \]
\[ 2ix_1 + (2 - i)x_2 + x_3 = 0 \]

\[ [x_1, x_2, x_3] = s[-3 + 2i, 3i, 1] \]

Solve the following system of equations:

\[ ix_1 + 2x_2 = 9 \]
\[ 3x_1 + (1 + i)x_2 = 5 + 8i \]

Find the inverse of the matrix \[ \begin{bmatrix} i & 1 \\ 2 & 3i \end{bmatrix} \]
Gaussian Elimination

Give a parametric equation for all solutions to the homogeneous system:

\[ \begin{align*}
ix_1 & + x_2 + 2x_3 = 0 \\
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\end{align*} \]

\[ [x_1, x_2, x_3] = s[-3 + 2i, 3i, 1] \]

Solve the following system of equations:

\[ \begin{align*}
ix_1 & + 2x_2 = 9 \\
3x_1 & + (1 + i)x_2 = 5 + 8i
\end{align*} \]

\[ x_1 = i, \ x_2 = 5 \]

Find the inverse of the matrix \[
\begin{bmatrix}
i & 1 \\
2 & 3i
\end{bmatrix}
\]
Gaussian Elimination

Give a parametric equation for all solutions to the homogeneous system:

\[
\begin{align*}
ix_1 + x_2 + 2x_3 &= 0 \\
ix_2 + 3x_3 &= 0 \\
2ix_1 + (2 - i)x_2 + x_3 &= 0
\end{align*}
\]

\([x_1, x_2, x_3] = s[-3 + 2i, 3i, 1]\)

Solve the following system of equations:

\[
\begin{align*}
ix_1 + 2x_2 &= 9 \\
3x_1 + (1 + i)x_2 &= 5 + 8i
\end{align*}
\]

\(x_1 = i, x_2 = 5\)

Find the inverse of the matrix

\[
\begin{bmatrix}
i & 1 \\
2 & 3i
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{-3i}{5} & \frac{1}{5} \\
\frac{2}{5} & -\frac{1}{5}i
\end{bmatrix}
\]
Exponentials

What to do when $i$ is the power of a function?
Exponentials

What to do when $i$ is the power of a function?

Maclaurin (Taylor) Series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$
Exponentials

What to do when $i$ is the power of a function?
Maclaurin (Taylor) Series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots$$
Exponentials

What to do when $i$ is the power of a function?

Maclaurin (Taylor) Series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} \cdots$$
Exponentials

What to do when \( i \) is the power of a function?

Maclaurin (Taylor) Series:

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots
\]

\[
e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots
\]

\[
e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} \cdots
\]

\[
e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)
\]
What to do when \( i \) is the power of a function?

Maclaurin (Taylor) Series:

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots
\]

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\]

\[
\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\]

\[
e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots
\]

\[
= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + \cdots
\]

\[
= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)
\]
Exponentials

What to do when $i$ is the power of a function?

Maclaurin (Taylor) Series:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= \cos x + isin x$$
Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]
Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

\[ \frac{d}{dx} [e^{ax}] = ae^{ax}; \]
Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

\[
\frac{d}{dx}[e^{ax}] = ae^{ax}; \\
\frac{d}{dx}[e^{ix}] 
\]
Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

\[
\frac{d}{dx}[e^{ax}] = ae^{ax}; \\
\frac{d}{dx}[e^{ix}] = \frac{d}{dx}[\cos x + i \sin x]
\]
5.3: Complex Exponential

Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

\[
\frac{d}{dx}[e^{ax}] = ae^{ax};
\]

\[
\frac{d}{dx}[e^{ix}] = \frac{d}{dx}[\cos x + i \sin x]
\]

\[
= -\sin x + i \cos x = i^2 \sin x + i \cos x = i(\cos x + i \sin x) = ie^{ix}
\]
Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

\[
\frac{d}{dx}[e^{ax}] = ae^{ax}; \\
\frac{d}{dx}[e^{ix}] = \frac{d}{dx} [\cos x + i \sin x] \\
= - \sin x + i \cos x = i^2 \sin x + i \cos x = i(\cos x + i \sin x) = ie^{ix}
\]

\[ e^{x+y} = e^x e^y; \]
Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

\[
\frac{d}{dx}[e^{ax}] = ae^{ax};
\]
\[
\frac{d}{dx}[e^{ix}] = \frac{d}{dx} [\cos x + i \sin x]
\]
\[= -\sin x + i \cos x = i^2 \sin x + i \cos x = i(\cos x + i \sin x) = ie^{ix} \]

\[ e^{x+y} = e^x e^y; \]
\[ e^{ix+iy} = \]
Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

\[
\frac{d}{dx} [e^{ax}] = ae^{ax}; \\
\frac{d}{dx} [e^{ix}] = \frac{d}{dx} [\cos x + i \sin x] \\
= -\sin x + i \cos x = i^2 \sin x + i \cos x = i(\cos x + i \sin x) = ie^{ix}
\]

\[ e^{x+y} = e^x e^y; \]
\[ e^{ix+iy} = e^{i(x+y)} = \cos(x + y) + i \sin(x + y) \]
Does that even make sense?

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\[ e^{x+y} = e^x e^y; \]
\[ e^{ix+iy} = e^{i(x+y)} = \cos(x+y) + i \sin(x+y) \]
\[ = \cos x \cos y - \sin x \sin y + i[\sin x \cos y + \cos x \sin y] \]
\[ = (\cos x + i \sin y)(\cos y + i \sin x) = e^{ix} e^{iy} \]
Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

Simplify:

\[ e^{\frac{\pi i}{2}} \]

\[ e^{2+i} \]

\[ \sqrt{2} e^{\frac{\pi i}{4}} \]

\[ 2^i \]

\[ e^{\pi i} + 1 \]
Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

Simplify:

\[ e^{\frac{\pi i}{2}} = i \]

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\[ 2^i = e^{i \ln 2} = \cos(\ln 2) + i \sin(\ln 2) \]

\[ e^{\pi i} + 1 = 0 \text{ (Euler’s Identity)} \]
Complex exponentiation: $e^{ix} = \cos x + i \sin x$

Let $x$ be a real number.
True or False?

1. $e^x = \cos x$
2. $e^{ix} = e^{i(x+2\pi)}$
3. $e^{ix} = -e^{i(x+\pi)}$
4. $e^{ix} + e^{-ix}$ is a real number
Coordinates Revisited

Complex number: $re^{i\theta} = re^{i(\theta + 2\pi)}$
Coordinates Revisited

\[ \begin{aligned} \text{Complex number:} & \quad r (\cos \theta + i \sin \theta) = re^{i\theta} \\ & \quad r (\cos (\theta + 2\pi) + i \sin (\theta + 2\pi)) \end{aligned} \]
Coordinates Revisited

Complex number: \( r \cos \theta + i \sin \theta \) = \( re^{i\theta} \) = \( re^{i(\theta + 2\pi)} \)
Coordinates Revisited

A complex number can be represented in polar form as $r \cos \theta + ir \sin \theta$, or equivalently, $re^{i\theta}$. The angle $\theta$ is measured from the positive real axis, and $r$ is the magnitude of the vector. The diagram illustrates this concept with a vector $\overrightarrow{OR}$, where $O$ is the origin, $R$ is a point on the plane, and $\theta$ is the angle from the positive real axis to the vector. The coordinates of $R$ can be expressed as $(r \cos \theta, r \sin \theta)$. The magnitude $r$ and angle $\theta$ provide a compact representation of the vector in the complex plane.
Coordinates Revisited

Complex number: \( r(\cos \theta + i \sin \theta) = re^{i\theta} \)
Coordinates Revisited

Complex number: \( r(\cos \theta + i \sin \theta) = re^{i\theta} \)
Coordinates Revisited

Complex number: \( r \cos \theta + i \sin \theta = re^{i\theta} = re^{i(\theta + 2\pi)} \)
Coordinates Revisited

Geometric interpretation of multiplication of two complex numbers:

Add the angles, multiply the lengths (moduli).

\[ re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta + \phi)} \]
Coordinates Revisited

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Coordinates Revisited

Geometric interpretation of multiplication of two complex numbers:
add the angles, multiply the lengths (moduli).
Roots of Unity

\[ e^{i \frac{2\pi}{n}} \]

for \( n = 1, 2, 3, \ldots \)
Roots of Unity

\[(re^{i\theta})^3 = 1\]
Roots of Unity

$e^{i \frac{2\pi}{3}}$

$e^{i \frac{4\pi}{3}}$

$(re^{i\theta})^3 = 1$
5.1: Complex Arithmetic
5.2: Complex Matrices and Linear Systems
5.3: Complex Exponential
5.4: Polar Representation

Roots of Unity

\[(re^{i\theta})^5 = 1\]
Roots of Unity

\[(re^{i\theta})^5 = 1\]
Roots of Unity

\[(re^{i\theta})^{12} = 1\]
Roots of Unity

\[(re^{i\theta})^{12} = 1\]
Find all complex numbers $z$ such that $z^3 = 8$.

Find all complex numbers $z$ such that $z^3 = 27 e^{i\pi 2}$.

Find all complex numbers $z$ such that $z^4 = 81 e^{2i}$. 
$z^3 = 8$

$2e^{\frac{2\pi i}{3}}$
$z^3 = 8$
$z^3 = 8$
$z^3 = 8$

$2e^{\frac{4\pi i}{3}}$
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