Outline

Week 8: Inverses and determinants

Course Notes: 4.5, 4.6

Goals: Be able to calculate a matrix's inverse; understand the relationship between the invertibility of a matrix and the solutions of associated linear systems; calculate the determinant of a square matrix of any size, and learn some tricks to make the computation more efficient.

Identity Matrix

The identity matrix, \( I \), is a square matrix with 1s along its main diagonal, and 0s everywhere else.

For any matrix \( A \) that can be multiplied with \( I \), \( AI = IA = A \).
What is Division?

\[(a + 5)x = 7x\]

Divide both sides by \(x\) **as long as** \(x \neq 0\)

There are some numbers we can’t divide by:

\[
\frac{(a + 5)x}{x} = 7x
\]

To divide by \(x\), we multiply by a special number (in this case, \(1/x\))

that has the following property: \(x(1/x)\) gives the multiplicative

identity.

\[
(a + 5)(1) = 7(1)
\]

1 is the multiplicative identity: If you multiply it by a number, that

number doesn’t change.

\[(a + 5) = 7\]

Matrix Inverses: The Closest we can Get to Division

Linear System Setup:

\[
\begin{align*}
  x + 2y + 3z &= 10 \\
  4x + 5y + 6z &= 20 \\
  7x + 8y + 9z &= 30 
\end{align*}
\]

\[
A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad b = \begin{bmatrix} 10 \\ 20 \\ 30 \end{bmatrix}
\]

\[
Ax = b
\]

Solve for \(x\).
Definition

A matrix $A^{-1}$ is the inverse of a square matrix $A$ if $A^{-1}A = I$, where $I$ is the identity matrix. In this case, also $AA^{-1} = I$.

What do you think the inverse of the following matrix should be?

$$\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}$$

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$$\begin{pmatrix}
\cos \theta & \sin \theta \\
\sin \theta & -\cos \theta
\end{pmatrix}$$

Existence of Matrix Inverses

Definition

A matrix $A^{-1}$ is the inverse of a square matrix $A$ if

$$A^{-1}A = I$$

where $I$ is the identity matrix.

Find the inverses of the following matrices:

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

If $Ax = b$ and $A^{-1}$ exists, then $x = A^{-1}b$.

If $A^{-1}$ exists, then $Ax = b$ has a unique solution.
If an Inverse Exists....

**Theorem**

If an \( n \times n \) matrix \( A \) has an inverse \( A^{-1} \), then for any \( b \) in \( \mathbb{R}^n \),

\[
Ax = b
\]

has precisely one solution, and that solution is \( x = A^{-1}b \).

So, if \( Ax = b \) has no solutions:

If \( Ax = b \) has infinitely many solutions:

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**Solutions to Systems of Equations**

Let \( A \) be an \( n \times n \) matrix. The following statements are equivalent:

1) \( Ax = b \) has exactly one solution for any \( b \).
2) \( Ax = 0 \) has no nonzero solutions.
3) The rank of \( A \) is \( n \).
4) The reduced form of \( A \) has no zeroes along the main diagonal.

By previous theorem, if \( A \) is invertible, all these statements hold.

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**If \( Ax = b \) has a unique solution for every \( b \), is \( A \) invertible?**

If \( T^{-1} \) is a linear transformation, then we can find a matrix \( B \) such that

\[
T^{-1}(b) = Bb
\]

for every \( b \). Then: \( x = Bb = B(Ax) = (BA)x \), so \( BA = I \).
If $Ax = b$ has a unique solution for every $b$, is $A$ invertible?

Need to show: $T^{-1}$ is a linear transformation.

- Fix $A$.
- Given $b$, we can solve $Ax = b$ for $x$.
- So, given $b$, we find $x$.
- This is a transformation: $T^{-1}(b) = x$. That is, given input $b$, the output $x$ is the vector we multiply $A$ by to get $b$.
- $T^{-1}$ is linear:
  - Let $T^{-1}(b_1) = x_1$ and $T^{-1}(b_2) = x_2$.
  - Note $A(x_1 + x_2) = Ax_1 + Ax_2 = b_1 + b_2$.
    So, $T^{-1}(b_1 + b_2) = x_1 + x_2 = T^{-1}(b_1) + T^{-1}(b_2)$.
    So, $T^{-1}$ preserves addition.
  - Note $A(sx_1) = s(Ax_1) = s(b_1)$, so $T^{-1}(sb_1) = sx_1 = sT^{-1}(b_1)$.
    So, $T^{-1}$ preserves scalar multiplication.
- Since $T^{-1}$ is a linear transformation from one collection of vectors to another, there exists some matrix $B$ such that $T^{-1}(b) = Bb$.
- Consider $T^{-1}(Ax)$. Note $T^{-1}(Ax) = x$ for every $x$ in $\mathbb{R}^n$, so $B(Ax) = x$ for every $x$. Therefore, $BA = I$, so $B = A^{-1}$.

Notes
An observation that will help compute inverses

Elementary row operations are equivalent to matrix multiplication.

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

More Conveniently Computing the Inverse (when it exists)

$$\begin{bmatrix} A & I \end{bmatrix} \xrightarrow{\text{reduce}} \begin{bmatrix} I & A^{-1} \end{bmatrix}$$

Calculate the inverse of

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 6 \\ 2 & 0 & 7 \end{bmatrix}$$

Calculate the inverse of

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
Using Inverses

Suppose \( M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \). Then (as we just found) \( M^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ 0 & 1 \end{bmatrix} \).
If \( Mx = \begin{bmatrix} 5 \\ 1 \end{bmatrix} \), what is \( x \)?

Suppose \( A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 1 \end{bmatrix} \) and \( A^{-1} = \begin{bmatrix} 7 & 0 & -3 \\ -2 & 1 & 0 \end{bmatrix} \).
If \( BA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \), what is \( B \)?

Inverses and Products

Suppose \( A \) and \( B \) are invertible matrices, with the same dimensions. Simplify:
\[
ABB^{-1}A^{-1}
\]
What is \((ABC)^{-1}\)?

Simplify:
\[
[(AC)^{-1}A(AB)^{-1}]^{-1}
\]

Determinants

Recall:
\[
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc
\]

In general:
\[
\det \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} = a_{1,1}D_{1,1} - a_{1,2}D_{1,2} + a_{1,3}D_{1,3} - \cdots \pm a_{1,n}D_{1,n}
\]
where \( D_{i,j} \) is the determinant of the matrix obtained from \( A \) by deleting row \( i \) and column \( j \).
Calculate

\[
\begin{vmatrix}
1 & 2 & 3 & 4 \\
0 & 1 & 0 & 1 \\
2 & 0 & 1 & 0 \\
1 & 0 & 2 & 0 \\
\end{vmatrix}
= \\
\begin{vmatrix}
0 & 1 & 0 & 1 \\
1 & 5 & 0 & 2 \\
2 & 0 & 5 & 1 \\
0 & 1 & 3 & 1 \\
\end{vmatrix}

Determinants of Triangular Matrices

Calculate, where \( \ast \) is any number:

\[
\begin{vmatrix}
1 & 0 & 0 & 0 \\
\ast & 2 & 0 & 0 \\
\ast \ast & 3 & 0 & 0 \\
\ast \ast \ast & 4 & 0 \\
\ast \ast \ast \ast & 5 \\
\end{vmatrix}

\[
\begin{vmatrix}
1 & \ast & \ast & \ast \\
0 & 2 & \ast & \ast \\
0 & 0 & 3 & \ast \\
0 & 0 & 0 & 4 \\
\end{vmatrix}

Fact: for any square matrix \( A \), \( \det(A) = \det(A^T) \)

Determinants of Upper Triangular Matrices

Is the determinant of ANY triangular matrix the product of the diagonal entries?
Careful: this ONLY works with triangular matrices!

**More Determinant Tricks**

Helpful Facts for Calculating the Determinant of a Square Matrix $A$:

1. If $B$ is obtained from $A$ by multiplying **one row** of $A$ by the constant $c$ then $\det B = c \det A$.
2. If $B$ is obtained from $A$ by **switching** two rows of $A$ then $\det B = -\det A$.
3. If $B$ is obtained from $A$ by **adding a multiple of one row** to another then $\det B = \det A$.
4. $\det(A) = 0$ if and only if $A$ is not invertible.
5. For all matrices $B$ of the same size as $A$, $\det(AB) = \det(A)\det(B)$.
6. $\det(A^T) = \det(A)$.

Remark: You should understand how the first three lead to the fourth; otherwise, the proofs are optional, found in the notes.

**If $A$ is invertible, then $\det(A) \neq 0$**

- $A$ is invertible
- we can row-reduce $A$ to the identity matrix
- we can row-reduce $A$ to a matrix with determinant 1
  - Adding a multiple of one row to another row does not change the determinant
  - Swapping two rows multiplies the determinant by $-1$
  - Multiplying a row by a constant $a$ multiplies the determinant by $a$
- $c \det(A) = 1$, where $c$ is some constant
- $\det(A) \neq 0$
Solutions to Systems of Equations

Let $A$ be an $n$-by-$n$ matrix. The following statements are equivalent:
1) $Ax = b$ has exactly one solution for any $b$.
2) $Ax = 0$ has no nonzero solutions.
3) The rank of $A$ is $n$.
4) The reduced form of $A$ has no zeroes along the main diagonal.
5) $A$ is invertible.
6) $\det(A) \neq 0$

Is $A$ invertible?

$A = \begin{bmatrix} 72 & 9 & 8 & 16 \\ 0 & 4 & 3 & -9 \\ 0 & 0 & 5 & 3 \\ 0 & 0 & 0 & 21 \end{bmatrix}$

$\det \begin{bmatrix} 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210$; $\det \begin{bmatrix} 0 & 1 & 2 & 0 \\ 10 & 0 & 1 \\ 10 & 0 & 5 & 3 \\ 0 & 2 & 1 & 1 \end{bmatrix} = ?$

Calculate:

$\det \begin{bmatrix} 1 & 5 & 10 & 15 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

$\det \begin{bmatrix} 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} = -210$; $\det \begin{bmatrix} 0 & 20 & 20 & 0 \\ 1 & 5 & 0 & 2 \\ 2 & 0 & 5 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix}$

$\det \begin{bmatrix} 2 & 0 & 5 & 1 \\ 1 & 5 & 0 & 2 \\ 0 & 10 & 10 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$

$\det \begin{bmatrix} 2 & 0 & 5 & 1 \\ 0 & 10 & 10 & 0 \\ 1 & 5 & 0 & 2 \\ 0 & 1 & 3 & 1 \end{bmatrix}$
Suppose det $A = 5$ for an invertible matrix $A$. What is $\det(A^{-1})$?

Suppose $A$ is an $n$-by-$n$ matrix with determinant 5. What is the determinant of $3A$?

Suppose $A$ is an $n$-by-$n$ matrix, and $x$ and $y$ are distinct vectors in $\mathbb{R}^n$ with $Ax = Ay$. What is $\det(A)$?

Using Row Reduction to Calculate a Determinant

$$\begin{pmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 4 \\ 1 & 2 & 8 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix}$$

$\det(\begin{pmatrix} 2 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 5 & 8 \end{pmatrix}) = \det(\begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 3 & 5 & 8 \end{pmatrix}) = -(-1) \det(\begin{pmatrix} 1 & 1 & 1 \\ 3 & 5 & 8 \\ 9 & 6 & 1 \end{pmatrix})$

$= -\det(\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 0 & -3 & -8 \end{pmatrix}) = \det(\begin{pmatrix} 1 & 2 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}) = 1 \det(\begin{pmatrix} 2 & 5 \\ -3 & -8 \end{pmatrix})$

$= -16 + 15 = -1$

Is the original 4-by-4 matrix invertible?
Determinant Expansion across Alternate Lines

“Line” means “row or column”

\[
\begin{vmatrix}
+ & - & + & - \\
- & + & - & + \\
+ & - & + & - \\
- & + & - & + \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
9 & 8 & 5 & 6 & 10 \\
1 & 0 & 0 & 0 & 1 \\
7 & 0 & 1 & 1 & 1 \\
8 & 0 & 1 & 1 & 1 \\
4 & 3 & 5 & 6 & 7 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
8 & 9 & 5 & 6 \\
0 & 1 & 1 & 0 \\
0 & 7 & 1 & 1 \\
0 & 8 & 1 & 1 \\
\end{vmatrix}
\]

More practice

\[
\begin{vmatrix}
2 & 5 & 3 & 4 \\
0 & 1 & 2 & 0 \\
4 & 4 & 6 & 9 \\
10 & 5 & 7 & 4 \\
\end{vmatrix}
\]