Outline

Week 7: Rotations, projections and reflections in 2D; matrix representation and composition of linear transformations; random walks; transpose.

Course Notes: 4.2, 4.3, 4.4

Goals: Understand that a linear transformation of a vector can always be achieved by matrix multiplication; use specific examples of linear transformations.
For a fixed vector $\mathbf{a}$, let $T(\mathbf{x}) = \text{proj}_a \mathbf{x}$.
For a fixed vector \( \mathbf{a} \), let \( T(\mathbf{x}) = \text{proj}_\mathbf{a} \mathbf{x} \)
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For a fixed vector \( \mathbf{a} \), let \( T(\mathbf{x}) = \text{proj}_\mathbf{a} \mathbf{x} \)
Projections

For a fixed vector \( \mathbf{a} \), let \( T(\mathbf{x}) = \text{proj}_\mathbf{a} \mathbf{x} \).
For a fixed vector $a$, let $T(x) = \text{proj}_a x$
For a fixed vector $a$, let $T(x) = \text{proj}_a x$
For a fixed vector $a$, let $T(x) = proj_a x$.
Computing Projections

Let \( \mathbf{a} = [a_1, a_2] \) and \( \mathbf{x} = [x_1, x_2] \).

\[
\text{proj}_\mathbf{a} \mathbf{x} = \frac{1}{a_1^2 + a_2^2} \begin{bmatrix} a_2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]
Computing Projections

Let \( \mathbf{a} = [a_1, a_2] \) and \( \mathbf{x} = [x_1, x_2] \).

\[
\text{proj}_\mathbf{a} \mathbf{x} = \frac{1}{a_1^2 + a_2^2} \begin{bmatrix} a_2^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

Let \( \mathbf{a} = [1, 1] \) and \( \mathbf{x} = [2, 3] \). Calculate \( \text{proj}_\mathbf{a} \mathbf{x} \) two ways.
Computing Projections

Let \( \mathbf{a} = [a_1, a_2] \) and \( \mathbf{x} = [x_1, x_2] \).

\[
\text{proj}_\mathbf{a} \mathbf{x} = \frac{1}{a_1^2 + a_2^2} \begin{bmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

Let \( \mathbf{a} = [1, 1] \) and \( \mathbf{x} = [2, 3] \). Calculate \( \text{proj}_\mathbf{a} \mathbf{x} \) two ways.

\[
T(\mathbf{x}) = \text{proj}_\mathbf{b} \left( \text{proj}_\mathbf{a} \mathbf{x} \right)
\]

Is the projection of a projection a projection? (Is there a vector \( \mathbf{c} \) so that \( T(\mathbf{x}) = \text{proj}_\mathbf{c} \mathbf{x} \)?)

Example: \( \mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \)
Reflections

For a fixed vector \( \mathbf{a} \), let \( \text{Ref}(\mathbf{x}) \) be the reflection of \( \mathbf{x} \) across the line through the origin in the direction of \( \mathbf{a} \).
Reflections

For a fixed vector $\mathbf{a}$, let $\text{Ref}(\mathbf{x})$ be the reflection of $\mathbf{x}$ across the line through the origin in the direction of $\mathbf{a}$. 

\[
\text{Ref}(\mathbf{x}) = \mathbf{x} + 2(\text{proj}_\mathbf{a}\mathbf{x} - \mathbf{x}) = 2\text{proj}_\mathbf{a}\mathbf{x} - \mathbf{x}
\]
For a fixed vector $\mathbf{a}$, let $\text{Ref}(\mathbf{x})$ be the reflection of $\mathbf{x}$ across the line through the origin in the direction of $\mathbf{a}$.
Reflections

For a fixed vector \( \mathbf{a} \), let \( \text{Ref}(\mathbf{x}) \) be the reflection of \( \mathbf{x} \) across the line through the origin in the direction of \( \mathbf{a} \).
Reflections

For a fixed vector $\mathbf{a}$, let $\text{Ref}(\mathbf{x})$ be the reflection of $\mathbf{x}$ across the line through the origin in the direction of $\mathbf{a}$.

\[
\text{Ref}(\mathbf{x}) = \mathbf{x} + 2(\text{proj}_a \mathbf{x} - \mathbf{x}) = 2\text{proj}_a \mathbf{x} - \mathbf{x}
\]
Reflections

For a fixed vector $\mathbf{a}$, let $\text{Ref}(\mathbf{x})$ be the reflection of $\mathbf{x}$ across the line through the origin in the direction of $\mathbf{a}$.

\[
\text{Ref}(\mathbf{x}) = \mathbf{x} + 2(\text{proj}_a \mathbf{x} - \mathbf{x}) = 2\text{proj}_a \mathbf{x} - \mathbf{x}
\]
Reflections

\[ \text{Ref}(x) = 2 \text{proj}_a x - x \]
Reflections

\[ \text{Ref} (\mathbf{x}) = 2\text{proj}_a \mathbf{x} - \mathbf{x} \]

Projections:

\[ \text{proj}_a \mathbf{x} = \frac{1}{a_1^2 + a_2^2} \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

Identity:

\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]
Reflections

\[ \text{Ref}(x) = 2\text{proj}_a x - x \]

Projections:

\[ \text{proj}_a x = \frac{1}{a_1^2 + a_2^2} \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

Identity:

\[ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ \text{Ref}(x) = 2\text{proj}_a x - x \]

\[ = \begin{bmatrix} \frac{2a_1^2}{a_1^2 + a_2^2} - 1 & \frac{2a_1 a_2}{a_1^2 + a_2^2} \\ \frac{2a_1 a_2}{a_1^2 + a_2^2} & \frac{2a_2^2}{a_1^2 + a_2^2} - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]
If $a$ is a unit vector, then $a_1^2 + a_2^2 = 1$. Then:

$$Ref(a) = \begin{bmatrix} \frac{2a_1^2}{a_1^2 + a_2^2} - 1 & \frac{2a_1a_2}{a_1^2 + a_2^2} \\ \frac{2a_1a_2}{a_1^2 + a_2^2} & \frac{2a_2^2}{a_1^2 + a_2^2} - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
Cleanup

\[
Ref(x) = \begin{bmatrix}
\frac{2a_1^2}{a_1^2 + a_2^2} - 1 & \frac{2a_1a_2}{a_1^2 + a_2^2} \\
\frac{2a_1a_2}{a_1^2 + a_2^2} & \frac{2a_2^2}{a_1^2 + a_2^2} - 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

If \(a\) is a unit vector, then \(a_1^2 + a_2^2 = 1\). Then:

\[
Ref(x) = \begin{bmatrix}
2a_1^2 - 1 & 2a_1a_2 \\
2a_1a_2 & 2a_2^2 - 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]
If \( a \) is a unit vector, then \( a_1^2 + a_2^2 = 1 \). Then:

\[
Ref(x) = \begin{bmatrix}
\frac{2a_1^2}{a_1^2+a_2^2} - 1 & \frac{2a_1a_2}{a_1^2+a_2^2} \\
\frac{2a_1a_2}{a_1^2+a_2^2} & \frac{2a_2^2}{a_1^2+a_2^2} - 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

And if \( a \) makes angle \( \theta \) with the \( x \)-axis, then \( a_1 = \cos \theta \) and \( a_2 = \sin \theta \), so:

\[
Ref_\theta(x) = \begin{bmatrix}
2a_1^2 - 1 & 2a_1a_2 \\
2a_1a_2 & 2a_2^2 - 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

\[
\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad \sin 2\theta = 2 \sin \theta \cos \theta
\]
**Cleanup**

\[
\text{Ref}(\mathbf{x}) = \begin{bmatrix}
\frac{2a_1^2}{a_1^2+a_2^2} - 1 & \frac{2a_1a_2}{a_1^2+a_2^2} \\
\frac{2a_1a_2}{a_1^2+a_2^2} & \frac{2a_2^2}{a_1^2+a_2^2} - 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

If \(\mathbf{a}\) is a unit vector, then \(a_1^2 + a_2^2 = 1\). Then:

\[
\text{Ref}(\mathbf{x}) = \begin{bmatrix}
2a_1^2 - 1 & 2a_1a_2 \\
2a_1a_2 & 2a_2^2 - 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

And if \(\mathbf{a}\) makes angle \(\theta\) with the \(x\)-axis, then \(a_1 = \cos \theta\) and \(a_2 = \sin \theta\), so:

\[
\text{Ref}_\theta(\mathbf{x}) = \begin{bmatrix}
\cos(2\theta) & \sin(2\theta) \\
\sin(2\theta) & -\cos(2\theta)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

\[
\cos^2 \theta = \frac{1 + \cos 2\theta}{2} \quad \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \quad \sin 2\theta = 2 \sin \theta \cos \theta
\]
Reflections and Rotations

Compare:

$$\text{Ref}_{\theta}(x) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{Rot}_{\phi}(x) = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
Reflections and Rotations

Compare:

\[
\text{Ref}_\theta(x) = \begin{bmatrix}
\cos(2\theta) & \sin(2\theta) \\
\sin(2\theta) & -\cos(2\theta)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

\[
\text{Rot}_\phi(x) = \begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

\[
\text{Ref}_\theta(x) \quad \text{Rot}_\phi(x)
\]
Reflections and Rotations

Compare:

$$\text{Ref}_\theta(x) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{Rot}_\phi(x) = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
Reflections and Rotations

Compare:

\[ \text{Ref}_\theta(x) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]

\[ \text{Rot}_\phi(x) = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \]
Reflections

To reflect $\mathbf{x}$ across the line through the origin that makes angle $\theta$ with the $x$-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
Reflections

To reflect \( \mathbf{x} \) across the line through the origin that makes angle \( \theta \) with the \( x \)-axis:

\[
\text{Ref}_\theta(\mathbf{x}) = \begin{bmatrix}
\cos(2\theta) & \sin(2\theta) \\
\sin(2\theta) & -\cos(2\theta)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

Example: find the reflection of the vector \([2, 4]\) across the line through the origin that makes an angle of 15 degrees with the \( x \)-axis.
Reflections

To reflect $\mathbf{x}$ across the line through the origin that makes angle $\theta$ with the $x$-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example: find the reflection of the vector $[2, 4]$ across the line through the origin that makes an angle of 15 degrees with the $x$-axis.

$$\begin{bmatrix} \cos(2(\pi/12)) & \sin(2(\pi/12)) \\ \sin(2(\pi/12)) & -\cos(2(\pi/12)) \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \cos(\pi/6) & \sin(\pi/6) \\ \sin(\pi/6) & -\cos(\pi/6) \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \sqrt{3} + 2 \\ 1 - 2\sqrt{3} \end{bmatrix} \approx \begin{bmatrix} 3.7 \\ -2.5 \end{bmatrix}$$
Reflections

To reflect $\mathbf{x}$ across the line through the origin that makes angle $\theta$ with the $x$-axis:

$$
\text{Ref}_\theta(\mathbf{x}) = \begin{bmatrix}
\cos(2\theta) & \sin(2\theta) \\
\sin(2\theta) & -\cos(2\theta)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
$$

What happens when we do two reflections?
Reflections

To reflect $\mathbf{x}$ across the line through the origin that makes angle $\theta$ with the $x$-axis:

$$Ref_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

What happens when we do two reflections?

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2\theta) \cos(2\phi) + \sin(2\theta) \sin(2\phi) & \cos(2\theta) \sin(2\phi) - \sin(2\theta) \cos(2\phi) \\ \sin(2\theta) \cos(2\phi) - \cos(2\theta) \sin(2\phi) & \sin(2\theta) \sin(2\phi) + \cos(2\theta) \cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2(\theta - \phi)) & -\sin(2(\theta - \phi)) \\ \sin(2(\theta - \phi)) & \cos(2(\theta - \phi)) \end{bmatrix} = Rot_{2(\theta-\phi)}$$
Two Reflections gives a Rotation

Consider:

- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.
Two Reflections gives a Rotation

Consider:

- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.
Two Reflections gives a Rotation

Consider:

- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.
Two Reflections gives a Rotation

Consider:
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.
Two Reflections gives a Rotation

Consider:

- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.
Two Reflections gives a Rotation

Consider:

- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.
Two Reflections gives a Rotation

Consider:

- Reflect across a line making an angle of $15°$ with the $x$-axis, then
- reflect across a line making an angle of $135°$ with the $x$-axis.
Two Reflections gives a Rotation

Consider:

- Reflect across a line making an angle of 15° with the x-axis, then
- Reflect across a line making an angle of 135° with the x-axis.
Two Reflections gives a Rotation

Consider:
- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.
Two Reflections gives a Rotation

Consider:

- Reflect across a line making an angle of $15^\circ$ with the $x$-axis, then
- Reflect across a line making an angle of $135^\circ$ with the $x$-axis.
Two Reflections gives a Rotation

Consider:

- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.
Two Reflections gives a Rotation

Consider:

- Reflect across a line making an angle of 15° with the x-axis, then
- reflect across a line making an angle of 135° with the x-axis.
Two Reflections gives a Rotation

Consider:

- Reflect across a line making an angle of 15° with the x-axis, then
- Reflect across a line making an angle of 135° with the x-axis.
Reflections

To reflect \( x \) across the line through the origin that makes angle \( \theta \) with the \( x \)-axis:

\[
Ref_\theta(x) = \begin{bmatrix}
\cos(2\theta) & \sin(2\theta) \\
\sin(2\theta) & -\cos(2\theta)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

What happens when we do two reflections?

\[
\begin{bmatrix}
\cos(2\theta) & \sin(2\theta) \\
\sin(2\theta) & -\cos(2\theta)
\end{bmatrix}
\begin{bmatrix}
\cos(2\phi) & \sin(2\phi) \\
\sin(2\phi) & -\cos(2\phi)
\end{bmatrix}
= \begin{bmatrix}
\cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) & \cos(2\theta)\sin(2\phi) - \sin(2\theta)\cos(2\phi) \\
\sin(2\theta)\cos(2\phi) - \cos(2\theta)\sin(2\phi) & \sin(2\theta)\sin(2\phi) + \cos(2\theta)\cos(2\phi)
\end{bmatrix}
= \begin{bmatrix}
\cos(2(\theta - \phi)) & -\sin(2(\theta - \phi)) \\
\sin(2(\theta - \phi)) & \cos(2(\theta - \phi))
\end{bmatrix}
= Rot_{2(\theta-\phi)}
\]

Are reflections commutative?

No (but almost)
Reflections

To reflect $\mathbf{x}$ across the line through the origin that makes angle $\theta$ with the $x$-axis:

$$
\text{Ref}_\theta(\mathbf{x}) = \begin{bmatrix}
\cos(2\theta) & \sin(2\theta) \\
\sin(2\theta) & -\cos(2\theta)
\end{bmatrix}
\begin{bmatrix}
\mathbf{x}_1 \\
\mathbf{x}_2
\end{bmatrix}
$$

What happens when we do two reflections?

$$
\begin{bmatrix}
\cos(2\theta) & \sin(2\theta) \\
\sin(2\theta) & -\cos(2\theta)
\end{bmatrix}
\begin{bmatrix}
\cos(2\phi) & \sin(2\phi) \\
\sin(2\phi) & -\cos(2\phi)
\end{bmatrix}
\begin{bmatrix}
\cos(2(\theta - \phi)) & -\sin(2(\theta - \phi)) \\
\sin(2(\theta - \phi)) & \cos(2(\theta - \phi))
\end{bmatrix}
= \text{Rot}_{2(\theta-\phi)}
$$

Are reflections commutative? No (but almost)
Reflections

To reflect $\mathbf{x}$ across the line through the origin that makes angle $\theta$ with the $x$-axis:

$$\text{Ref}_\theta(\mathbf{x}) = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

What happens when we do two reflections?

$$\begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} \cos(2\phi) & \sin(2\phi) \\ \sin(2\phi) & -\cos(2\phi) \end{bmatrix} = \begin{bmatrix} \cos(2\theta)\cos(2\phi) + \sin(2\theta)\sin(2\phi) & \cos(2\theta)\sin(2\phi) - \sin(2\theta)\cos(2\phi) \\ \sin(2\theta)\cos(2\phi) - \cos(2\theta)\sin(2\phi) & \sin(2\theta)\sin(2\phi) + \cos(2\theta)\cos(2\phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(2(\theta - \phi)) & -\sin(2(\theta - \phi)) \\ \sin(2(\theta - \phi)) & \cos(2(\theta - \phi)) \end{bmatrix} = \text{Rot}_{2(\theta - \phi)}$$

Are reflections commutative? No (but almost)

Are reflections commutative with rotations?
Reflections and Rotations

Are reflections commutative with rotations?

Try the following with a cell phone or book:
1. Rotate 90 degrees clockwise
2. Flip 180 degrees vertically

Alternately:
1. Flip 180 degrees vertically
2. Rotate 90 degrees clockwise
Reflections and Rotations

Are reflections commutative with rotations?

Try the following with a cell phone or book:
1. Rotate 90 degrees clockwise
2. Flip 180 degrees vertically

Alternately:
1. Flip 180 degrees vertically
2. Rotate 90 degrees clockwise

Nope.
Summary: Examples of Linear Transformations

To compute the rotation of the vector $\mathbf{x}$ by $\theta$, multiply $\mathbf{x}$ by the matrix

$$
Rot_\theta = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta \\
\end{bmatrix}
$$
Summary: Examples of Linear Transformations

To compute the rotation of the vector \( x \) by \( \theta \), multiply \( x \) by the matrix

\[
Rot_\theta = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

To compute the projection of the vector \( x \) onto the vector \( [a_1, a_2] \), multiply \( x \) by the matrix

\[
proj_{[a_1, a_2]} = \begin{bmatrix}
a_1^2 & a_1 a_2 \\
a_1^2 + a_2^2 & a_1^2 + a_2^2 \\
a_1 a_2 & a_2^2 \\
a_1^2 + a_2^2 & a_1^2 + a_2^2
\end{bmatrix}
\]
Summary: Examples of Linear Transformations

To compute the rotation of the vector $x$ by $\theta$, multiply $x$ by the matrix

$$\text{Rot}_\theta = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

To compute the projection of the vector $x$ onto the vector $[a_1, a_2]$, multiply $x$ by the matrix

$$\text{proj}_{[a_1, a_2]} = \begin{bmatrix} \frac{a_1^2}{a_1^2 + a_2^2} & \frac{a_1 a_2}{a_1^2 + a_2^2} \\ \frac{a_1 a_2}{a_1^2 + a_2^2} & \frac{a_2^2}{a_1^2 + a_2^2} \end{bmatrix}$$

To compute the reflection of the vector $x$ across the line through the origin that makes an angle of $\phi$ with the $x$-axis, multiply $x$ by the matrix

$$\text{Ref}_\phi = \begin{bmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{bmatrix}$$
Summary: Examples of Linear Transformations

To compute the rotation of the vector \( \mathbf{x} \) by \( \theta \), multiply \( \mathbf{x} \) by the matrix

\[
\text{Rot}_\theta = \begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\]

To compute the projection of the vector \( \mathbf{x} \) onto the vector \([a_1, a_2] \), multiply \( \mathbf{x} \) by the matrix

\[
\text{proj}_{[a_1, a_2]} = \begin{bmatrix}
a_1^2 & a_1 a_2 \\
\frac{a_1^2}{a_1^2+a_2^2} & \frac{a_1 a_2}{a_1^2+a_2^2} \\
\frac{a_2^2}{a_1^2+a_2^2} & \frac{a_2^2}{a_1^2+a_2^2}
\end{bmatrix}
\]

To compute the reflection of the vector \( \mathbf{x} \) across the line through the origin that makes an angle of \( \phi \) with the \( x \)-axis, multiply \( \mathbf{x} \) by the matrix

\[
\text{Ref}_\phi = \begin{bmatrix}
\cos 2\phi & \sin 2\phi \\
\sin 2\phi & -\cos 2\phi
\end{bmatrix}
\]

Which transformations are equivalent to matrix multiplication?
Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T([1,0])$ and $T([0,1])$.
- Since $\{[1,0],[0,1]\}$ is a basis of $\mathbb{R}^2$, every vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors. For example, $[9,14] = 9[1,0] + 14[0,1]$.
- Since $T$ is linear, $T([9,14]) = T(9[1,0] + 14[0,1]) = 9T([1,0]) + 14T([0,1])$.
- In general, $T([x,y]) = T(x[1,0] + y[0,1]) = xT([1,0]) + yT([0,1])$. 

4.3: Application: Random Walks

4.3: The Transpose
Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$. 
Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$
Which transformations are equivalent to matrix multiplication?

Suppose \( T \) is a transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^4 \).

- Suppose we know \( T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \) and \( T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \)

- Since \( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \) is a basis of \( \mathbb{R}^2 \), every vector in \( \mathbb{R}^2 \) can be written as a linear combination of these two vectors.
Which transformations are equivalent to matrix multiplication?

Suppose \( T \) is a transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^4 \).

- Suppose we know \( T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \) and \( T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \).

- Since \( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \) is a basis of \( \mathbb{R}^2 \), every vector in \( \mathbb{R}^2 \) can be written as a linear combination of these two vectors. For example, \( \begin{bmatrix} 9 \\ 14 \end{bmatrix} = 9 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 14 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \).
Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

- Since $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis of $\mathbb{R}^2$, every vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors.

  For example, $\begin{bmatrix} 9 \\ 14 \end{bmatrix} = 9 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 14 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Since $T$ is linear,

  $T\left(\begin{bmatrix} 9 \\ 14 \end{bmatrix}\right)$
Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $T \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

- Since $\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis of $\mathbb{R}^2$, every vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors.

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Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$ and $T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

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  For example, $\begin{bmatrix} 9 \\ 14 \end{bmatrix} = 9 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 14 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Since $T$ is linear, $T \left( \begin{bmatrix} 9 \\ 14 \end{bmatrix} \right) = T \left( 9 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 14 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = 9T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + 14T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$
Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

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- Since $T$ is linear, $T\left(\begin{bmatrix} 9 \\ 14 \end{bmatrix}\right) = T\left(9 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 14 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 9T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 14T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$.

- In general, $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.
Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

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- Since $T$ is linear,
  \[
  T\left(\begin{bmatrix} 9 \\ 14 \end{bmatrix}\right) = T\left(9 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 14 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = 9 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 14 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)
  \]

- In general,
  \[
  T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)
  \]
Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Since $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis of $\mathbb{R}^2$, every vector in $\mathbb{R}^2$ can be written as a linear combination of these two vectors. For example, $\begin{bmatrix} 9 \\ 14 \end{bmatrix} = 9 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 14 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- Since $T$ is linear, $T \begin{bmatrix} 9 \\ 14 \end{bmatrix} = T \left( 9 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 14 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = 9 T \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 14 T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- In general, $T \begin{bmatrix} x \\ y \end{bmatrix} = T \left( x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = x T \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y T \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) =$ and $T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) =$
Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$
Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$. 

- Suppose we know $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$.

- 

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 5 \\ 2 5 \\ 3 5 \\ 4 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$

- $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$
Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$.

- 
  
  $T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = T \left( x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = x T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + y T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

  $= x \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + y \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$
Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$.

- \[
T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)
\]

\[
= x\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + y\begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1x + 5y \\ 2x + 5y \\ 3x + 5y \\ 4x + 5y \end{bmatrix}
\]
Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

• Suppose we know $T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$

• $T \left( \begin{bmatrix} x \\ y \end{bmatrix} \right) = T \left( x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = x T \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + y T \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$

$$= x \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + y \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1x + 5y \\ 2x + 5y \\ 3x + 5y \\ 4x + 5y \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 2 & 5 \\ 3 & 5 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
Which transformations are equivalent to matrix multiplication?

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^4$.

- Suppose we know $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix}$

- $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = xT\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + yT\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

  $$= x\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + y\begin{bmatrix} 5 \\ 5 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 1x + 5y \\ 2x + 5y \\ 3x + 5y \\ 4x + 5y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- So: $T(x)$ can be computed as a matrix multiplication,

$$T(x) = \begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix} x$$
Which transformations are equivalent to matrix multiplication?

**Theorem**

Every linear transformation $T$ that takes a vector as an input, and gives a vector as an output, is equivalent to a matrix multiplication.
Which transformations are equivalent to matrix multiplication?

**Theorem**
Every linear transformation $T$ that takes a vector as an input, and gives a vector as an output, is equivalent to a matrix multiplication.

**Extended Theorem**
Suppose $T$ is a linear transformation that transforms vectors of $\mathbb{R}^n$ into vectors of $\mathbb{R}^m$. If $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$, then:

$$T \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

That is: $e_1 = [1, 0, \ldots, 0]$, $e_2 = [0, 1, 0, \ldots, 0]$, etc.
Geometric interpretation of an $n$-by-$m$ matrix: linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.

A matrix can be viewed as a particular kind of function.
General Linear Transformations

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}, \quad
\begin{bmatrix}
0 \\
0 \\
1
\end{bmatrix}
\]
General Linear Transformations

\[
\begin{aligned}
\{ & \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
& \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \\
\end{aligned}
\]

\[
T \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad T \left( \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \quad T \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}
\]
General Linear Transformations

\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \quad
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\]

\[
T \left( \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix} \right) = \begin{pmatrix} 1 \end{pmatrix}, \quad
T \left( \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix} \right) = \begin{pmatrix} 2 \end{pmatrix}, \quad
T \left( \begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix} \right) = \begin{pmatrix} 3 \end{pmatrix}
\]

\[
T \left( \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} \right) = x \begin{pmatrix} 1 \end{pmatrix} + y \begin{pmatrix} 2 \end{pmatrix} + z \begin{pmatrix} 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x \\
y \\
z \end{pmatrix}
\]
General Linear Transformations

\[ T : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad \text{linear} \]
General Linear Transformations

\[ T : \mathbb{R}^n \to \mathbb{R}^m \text{ linear} \]

Standard basis of \( \mathbb{R}^n \):

\[
\begin{align*}
\{ e_1 &= \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \\
e_2 &= \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \\
&\cdots, \\
e_n &= \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \}
\end{align*}
\]
General Linear Transformations

\[ T : \mathbb{R}^n \rightarrow \mathbb{R}^m \quad \text{linear} \]

Standard basis of \( \mathbb{R}^n \):

\[
\left\{
\begin{array}{c}
  e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \\
  e_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \\
  \ldots, \\
  e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}
\end{array}
\right.
\]

\[
T \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix}
  T(e_1) & T(e_2) & \cdots & T(e_n)
\end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\]
Examples

Suppose a linear transformation $T$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ has the following properties:

$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$

Give a matrix $A$ so that $T(x) = Ax$ for every vector $x$ in $\mathbb{R}^2$. 
Examples

Suppose a linear transformation $T$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ has the following properties:

\[
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\]

\[
T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}.
\]

Give a matrix $A$ so that $T(x) = Ax$ for every vector $x$ in $\mathbb{R}^2$.

Suppose a linear transformation $T$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ has the following properties:

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\]

\[
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\]

Give a matrix $A$ so that $T(x) = Ax$ for every vector $x$ in $\mathbb{R}^2$. 
Examples

Suppose $T$ is a transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$, where $T(x) = Ax$ for the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Which vector $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ has $T(x) = \begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}$?

Which vector $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ has $T(y) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$?
Examples

Suppose \( T \) is a transformation from \( \mathbb{R}^2 \) to \( \mathbb{R}^3 \), where \( T(x) = Ax \) for the matrix

\[
A = \begin{bmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{bmatrix}
\]

Which vector \( x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \) has \( T(x) = \begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix} \)?

Which vector \( y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \) has \( T(y) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \)?

Characterize vectors that can come out of \( T \).
Random Walks: Another Use of Matrix Multiplication

- $n$ states
- Fixed probability $p_{i,j}$ of moving to state $i$ if you are in state $j$. 

Examples: https://en.wikipedia.org/wiki/Random_walk
model Brownian Motion (Wiener process)
genetic drift
stock markets
use sampling to estimate properties of a large system
Random Walks: Another Use of Matrix Multiplication

- $n$ states
- Fixed probability $p_{i,j}$ of moving to state $i$ if you are in state $j$.

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- use sampling to estimate properties of a large system
Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

<table>
<thead>
<tr>
<th>from</th>
<th>sleeping</th>
<th>fishing</th>
<th>playing</th>
</tr>
</thead>
<tbody>
<tr>
<td>to</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sleeping</td>
<td>.5</td>
<td>.7</td>
<td>.4</td>
</tr>
<tr>
<td>fishing</td>
<td>.25</td>
<td>0</td>
<td>.3</td>
</tr>
<tr>
<td>playing</td>
<td>.25</td>
<td>.3</td>
<td>.3</td>
</tr>
</tbody>
</table>

Fishing: By Mimooh (Own work), via Wikimedia Commons
Playing: By Silvermoonlight217
Random Walks: Another Use of Matrix Multiplication

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<tr>
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Let $x_n$ be the vector describing the probability that the penguin is sleeping/fishing/playing after $n$ hours.
Random Walks: Another Use of Matrix Multiplication

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<td>.3</td>
<td>.3</td>
</tr>
</tbody>
</table>

Let $x_n$ be the vector describing the probability that the penguin is sleeping/fishing/playing after $n$ hours.

$x_0$: initial state of penguin. For example: $[1, 0, 0]$ if we know the penguin is sleeping.
Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

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Random Walks: Another Use of Matrix Multiplication

An ideal penguin has three states: sleeping, fishing, and playing. It is observed once per hour.

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Random Walks

In general:
- $n$ states
- $p_{i,j}$ probability of moving to state $i$ if you are in state $j$; $P = [p_{i,j}]$
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$P$: "transition matrix"
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Suppose you are learning to walk on a tight rope, but you are not very good yet. With every step you take, your chances of falling to the right are 1%, and your changes of falling to the left are 5%, because of an old math-related injury that causes you to lean left when you’re scared. When you fall, you stay on the ground.
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Where are you after 100 steps?
Random Walk Example: Error Messages

Suppose you are using a buggy program. You start up without a problem.

- If you have never encountered an error message, your odds of encountering an error message with your next click are 0.01.

- If you have already encountered exactly one error message, your odds of encountering a second on your next click are 0.05.

- If you have encountered two error messages, the odds of encountering a third on your next click are 0.1.

- After the third error message, you uninstall the program, and never use it again.
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Possible states: no errors; one error; two errors; three errors; uninstalled.
Random Walk Example

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Harder Questions involving Random Walks

- For which value of $n$ does $x_n$ have a certain characteristic?
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Note: $\lim_{n \to \infty} x_n = \lim_{n \to \infty} P^n x_0$. 

Application: Google!

Stay tuned for more Random Walks excitement.
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Application: Google!

Stay tuned for more Random Walks excitement
Transpose

Transpose: rows $\leftrightarrow$ columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$
Transpose: rows ↔ columns.

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \]

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\[ B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \]

\[ B^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \]


Transpose

Transpose: rows ↔ columns.

\[
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\end{bmatrix}
\]

\[
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3
\end{bmatrix}
\begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
1 & 2 & 3
\end{bmatrix}
\]

\[
B^T = \begin{bmatrix}
1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 3
\end{bmatrix}
\]

\[
AB = \begin{bmatrix}
6 & 12 & 18 \\
15 & 30 & 45
\end{bmatrix}
\]

\[
BA = DNE
\]
Transpose

Transpose: rows $\leftrightarrow$ columns.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 6 & 12 & 18 \\ 15 & 30 & 45 \end{bmatrix}$$

$$BA = DNE$$

$$B^T A^T = \begin{bmatrix} 6 & 15 \\ 12 & 30 \\ 18 & 45 \end{bmatrix}$$

$$AB = (B^T A^T)^T$$
Previous example of noncommutativity of matrix multiplication:

\[
\begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
7 & 5 \\
3 & 0
\end{bmatrix}
= 
\begin{bmatrix}
13 & 5 \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
7 & 5 \\
3 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}
= 
\begin{bmatrix}
7 & 14 \\
3 & 6
\end{bmatrix}
\]
Previous example of noncommutativity of matrix multiplication:

\[
\begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
7 & 5 \\
3 & 0
\end{bmatrix}
= 
\begin{bmatrix}
13 & 5 \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
7 & 5 \\
3 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}
= 
\begin{bmatrix}
7 & 14 \\
3 & 6
\end{bmatrix}
\]

\[
\begin{bmatrix}
7 & 3 \\
5 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
2 & 0
\end{bmatrix}
= 
\begin{bmatrix}
13 & 0 \\
5 & 0
\end{bmatrix}
\]
Transpose and Dot Product

\[ y \cdot (Ax) = (A^T y) \cdot x \]

where \( A \) is an \( m \)-by-\( n \) matrix, \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \).
Transpose and Dot Product

\[ y \cdot (Ax) = (A^T y) \cdot x \]

where \( A \) is an \( m \)-by-\( n \) matrix, \( x \in \mathbb{R}^n \) and \( y \in \mathbb{R}^m \).

\[
\begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix} \cdot \left( \begin{pmatrix}
1 & 0 \\
0 & 1 \\
-1 & 1
\end{pmatrix} \begin{pmatrix}
8 \\
9
\end{pmatrix} \right) = \begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix} \cdot \begin{pmatrix}
8 \\
9
\end{pmatrix} = 8 + 18 + 3 = 29
\]

\[
\begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 1
\end{pmatrix} \begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix} \cdot \begin{pmatrix}
8 \\
9
\end{pmatrix} = \begin{pmatrix}
-2 \\
5
\end{pmatrix} \cdot \begin{pmatrix}
8 \\
9
\end{pmatrix} = -16 + 45 = 29
\]
Summary

- Transpose swaps rows and columns
- $AB = (B^T A^T)^T$
- $y \cdot (Ax) = (A^T y) \cdot x$

- $(A^T)^T = A$
- $\left( \left( \left( (A^T)^T \right)^T \right)^T \right)^T = A$
- $(AB)x = (x^T B^T)^T A$
- $y \cdot (Ax) = x \cdot (A^T y)$
True or False?

Summary

- Transpose swaps rows and columns
- \( AB = (B^T A^T)^T \)
- \( y \cdot (Ax) = (A^T y) \cdot x \)

- \( (A^T)^T = A \) \( \text{true} \)

- \( \left( \left( \left( (A^T)^T \right)^T \right)^T \right)^T = A \)

- \( (AB)x = (x^T B^T)^T A \)

- \( y \cdot (Ax) = x \cdot (A^T y) \)
True or False?

Summary

• Transpose swaps rows and columns

• $AB = (B^T A^T)^T$

• $y \cdot (Ax) = (A^T y) \cdot x$

• $(A^T)^T = A$  \hspace{1cm} \text{true}

• $\left( \left( \left( (A^T)^T \right)^T \right)^T \right)^T = A$  \hspace{1cm} \text{false}

• $(AB)x = (x^T B^T)^T A$

• $y \cdot (Ax) = x \cdot (A^T y)$
Summary

• Transpose swaps rows and columns
• $AB = (B^T A^T)^T$
• $y \cdot (Ax) = (A^T y) \cdot x$

- $(A^T)^T = A$ \hspace{1cm} true
- $\left( \left( \left( (A^T)^T \right)^T \right)^T \right)^T = A$ \hspace{1cm} false
- $(AB)x = (x^T B^T)^T A$ \hspace{1cm} false
- $y \cdot (Ax) = x \cdot (A^T y)$
Summary

- Transpose swaps rows and columns
- \( AB = (B^T A^T)^T \)
- \( y \cdot (Ax) = (A^T y) \cdot x \)

- \( (A^T)^T = A \) \quad \text{true}
- \( 
\left( \left( \left( \left( (A^T)^T \right)^T \right)^T \right)^T \right)^T = A \) \quad \text{false}
- \( (AB)x = (x^T B^T)^T A \) \quad \text{false}
- \( y \cdot (Ax) = x \cdot (A^T y) \) \quad \text{true}