Week 6: Matrix Multiplication and Linear Transformation

Course Notes: 4.1, 4.2

Goals: Learn the mechanics of matrix multiplication and linear transformation, and use matrix multiplication to describe linear transformations.

Matrix Anatomy

A matrix with 3 rows and 4 columns is a 3 by 4 matrix.

We often write $A = [a_{i,j}]$, where $a_{i,j}$ refers to the particular entry of $A$ in row $i$, column $j$.

Addition and Scalar Multiplication

Addition and scalar multiplication work the way you want them to.

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 5 & -1 \\ 8 & 6 & 6 & 2 \\ 3 & -1 & 2 & -3 \end{bmatrix}$
You’re comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three services: texts, minutes talking, and GB of data.

You want to know, for each person and plan, what the cost will be.

Input: plans×services and people×services
Output: plans×people

Matrix Multiplication

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 \\
2 & 1 \\
0 & 3
\end{bmatrix}
= \begin{bmatrix}
5 & 11 \\
10 & 22
\end{bmatrix}
\]

In the product, the entry in the ith row and jth column comes from dotting the ith row and jth column of the matrices being multiplied.

\[
\begin{align*}
[1, 2, 3] \cdot [1, 2, 0] &= 5 \\
[1, 2, 3] \cdot [0, 1, 3] &= 11 \\
[2, 4, 6] \cdot [1, 2, 0] &= 10 \\
[2, 4, 6] \cdot [0, 1, 3] &= 22
\end{align*}
\]

Another Example

\[
\begin{bmatrix}
0 & 1 & 3 \\
1 & 0 & 2 \\
1 & 1 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
2 & 3 \\
3 & 0 \\
1 & 2
\end{bmatrix}
= \begin{bmatrix}
\end{bmatrix}
\]
Another Example

\[
\begin{bmatrix}
2 & 5 \\
0 & 1 \\
1 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 3 & 1 \\
0 & 1 & 1 & 1
\end{bmatrix}
= \]

Wait but... why

\[
\begin{bmatrix}
x_1 + 2x_2 + 3x_3 + 4x_4 \\
5x_1 + 6x_2 + 7x_3 + 8x_4
\end{bmatrix}
= \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\]

\[
A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\]

Dimensions

\[
\begin{bmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{bmatrix}
= \begin{bmatrix}
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & * \\
* & * & * & * & *
\end{bmatrix}
\]

We can only take the dot product of two vectors that have the same length.

If \( A \) is an \( m \)-by-\( n \) matrix, and \( B \) is an \( r \)-by-\( c \) matrix, then \( AB \) is only defined if \( n = r \). If \( n = r \), then \( AB \) is an \( m \)-by-\( c \) matrix.

Can you always multiply a matrix by itself?
Properties of Matrix Multiplication

One important property DOESN’T hold.

\[
\begin{bmatrix}
1 & 2 \\
0 & 0 \\
7 & 5 \\
3 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}
= 
\begin{bmatrix}
1 & 2 \\
0 & 0 \\
7 & 5 \\
3 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
1 & 2
\end{bmatrix}
\]

Properties of Matrix Algebra

The other properties hold as you would like. (Page 128, notes.)
1. \( A + B = B + A \)
2. \( A + (B + C) = (A + B) + C \)
3. \( s(A + B) = sA + sB \)
4. \( (s + t)A = sA + tA \)
5. \( (st)A = s(tA) \)
6. \( 1A = A \)
7. \( A + 0 = A \) (where \( 0 \) is the matrix of all zeros)
8. \( A - A = A + (-1)A = 0 \)
9. \( A(B + C) = AB + AC \)
10. \( (A + B)C = AC + BC \)
11. \( A(BC) = (AB)C \)
12. \( s(AB) = (sA)B = A(sB) \)

Examples

Simplify the following expressions.

1) \[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 3
\end{bmatrix}
\begin{bmatrix}
8 & 9 & 8 \\
9 & 8 & 9 \\
8 & 9 & 8
\end{bmatrix}
+ 
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 3
\end{bmatrix}
\begin{bmatrix}
-8 & -9 & -8 \\
-8 & -9 & -8 \\
-8 & -9 & -8
\end{bmatrix}
\]

2) \[
\begin{bmatrix}
33 & 44 \\
55 & 66
\end{bmatrix}
\begin{bmatrix}
5 & 1 \\
7 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
1 & 1
\end{bmatrix}
\]

3) \[
2.8 \begin{bmatrix}
15 & 0 & 38 \\
9 & 10 & 11 \\
8 & 7 & 6
\end{bmatrix}
+ 5.6 \begin{bmatrix}
-2.5 & 0 & 1 \\
0.5 & 0 & -0.5 \\
1 & 1.5 & 2
\end{bmatrix}
\]
Suppose $A$ is an $m$-by-$n$ matrix, and $B$ is an $r$-by-$c$ matrix.

If we want to multiply $A$ and $B$, what has to be true about $m$, $n$, $r$, and $c$?

If we want to add $A$ and $B$, what has to be true about $m$, $n$, $r$, and $c$?

If we want to compute $(A + B)A$, what has to be true about $m$, $n$, $r$, and $c$?

\[ f(x) = x^2 \]
\[ f(2 + 3) = 25 \quad f(2) + f(3) = 4 + 9 = 13 \]
\[ f(2 + 3) = 36 \quad 2f(3) = 2 \cdot 9 = 18 \]

\[ g(x) = 5x \]
\[ g(2 + 3) = 25 \quad g(2) + g(3) = 10 + 15 = 25 \]
\[ g(2 + 3) = 30 \quad 2g(3) = 2 \cdot 15 = 30 \]

\[ g(x + y) = 5(x + y) = 5x + 5y = g(x) + g(y) \]
\[ g(xy) = 5(xy) = x(5y) = xg(y) \]
Linear Transformations

Definition
A transformation $T$ is called linear if, for any $x, y$ in the domain of $T$, and any scalar $s$,

$$T(x + y) = T(x) + T(y)$$

and

$$T(sx) = sT(x).$$

Is differentiation $T(f(x)) = \frac{d}{dx}[f(x)]$ (of functions whose derivatives exist everywhere) a linear transformation?

Let $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + y \\ 2x \end{bmatrix}$. Is $T$ a linear transformation?

Are the following linear transformations?

$$S\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} z \\ y \\ x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$T(x) = |x|, \ x \in \mathbb{R}^2$$

$$R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -1 \\ y \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Is the transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ linear?

If $A$ is a matrix, then the transformation $T(x) = Ax$ of a vector $x$ is linear.
Geometric Interpretation

We interpret a matrix geometrically as a function from some vectors to some other vectors. In particular, the function is a linear transformation, so it preserves addition and scalar multiplication.

If \( T(x) = Ax \) for some \( 3 \times 5 \) matrix \( A \) (and a vector \( x \)), what are the domain and range of the function \( T \)?

Example

Let \( T(x) \) be the rotation of \( x \) by ninety degrees.

Rotation by a fixed angle is a linear transformation.

Computing a rotations of \( \phi \) radians (\( \phi \) fixed)
**Computing Rotations**

\[ \begin{align*}
|v| \sin(\theta + \phi) & \\
|v| \sin \theta & \\
|v| \cos(\theta + \phi) \cos \theta & \\
\end{align*} \]

\[ \mathbf{v} = [v_1, v_2]; \quad T(\mathbf{v}) = [x, y] \]

\[ 
\begin{align*}
x &= |v| \cos(\theta + \phi) \\
&= |v| (\cos \theta \cos \phi - \sin \phi \sin \theta) \\
&= v_1 \cos \phi - v_2 \sin \phi \\
y &= |v| \sin(\theta + \phi) \\
&= |v| (\sin \theta \cos \phi + \cos \theta \sin \phi) \\
&= v_1 \sin \phi + v_2 \cos \phi \\
\end{align*} \]

**Rot \( \phi \)**

\[
\begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}
\]

What matrix should you multiply \[ \begin{bmatrix} 4 \\ 2 \end{bmatrix} \] by to rotate it 90 degrees \((\pi/2 \text{ radians})\)?

What matrix should you multiply \[ \begin{bmatrix} 4 \\ 2 \end{bmatrix} \] by to rotate it 30 degrees \((\pi/6 \text{ radians})\)?

**Are rotations commutative?**

Let \( a \) be a vector in \( \mathbb{R}^2 \).

1. Rotate the vector \( a \) by \( \theta \) radians, then by \( \phi \) radians.
2. Rotate the vector \( a \) by \( \phi \) radians, then by \( \theta \) radians.

Will you always end up with the same thing?
Are rotations commutative?

Let \( a \) be a vector in \( \mathbb{R}^2 \).

1. Rotate the vector \( a \) by \( \theta \) radians, then by \( \phi \) radians.

2. Rotate the vector \( a \) by \( \phi \) radians, then by \( \theta \) radians.

Will you always end up with the same thing?

Will \( \text{Rot}_\phi (\text{Rot}_\theta a) = \text{Rot}_\theta (\text{Rot}_\phi a) \) for every \( \theta \), every \( \phi \), and every \( a \) in \( \mathbb{R}^2 \)?

In general, matrix multiplication is not commutative, but we don’t care about ALL matrices—only rotation matrices.