Outline

Week 6: Matrix Multiplication and Linear Transformation

Course Notes: 4.1, 4.2

Goals: Learn the mechanics of matrix multiplication and linear transformation, and use matrix multiplication to describe linear transformations.
A matrix with 3 rows and 4 columns is a **3 by 4** matrix.
A matrix with 3 rows and 4 columns is a 3 by 4 matrix.

We often write $A = [a_{i,j}]$, where $a_{i,j}$ refers to the particular entry of $A$ in row $i$, column $j$. 

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}$$
## Matrix Anatomy

A matrix with 3 rows and 4 columns is a **3 by 4 matrix**.

We often write $A = [a_{i,j}]$, where $a_{i,j}$ refers to the particular entry of $A$ in row $i$, column $j$.

Here, $a_{3,2} = 6$
Addition and scalar multiplication work the way you want them to.

\[ A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 5 & -1 \\ 8 & 6 & 6 & 2 \\ 3 & -1 & 2 & -3 \end{bmatrix} \]
Addition and Scalar Multiplication

Addition and scalar multiplication work the way you want them to.

\[ A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 5 & -1 \\ 8 & 6 & 6 & 2 \\ 3 & -1 & 2 & -3 \end{bmatrix} \]

\[ A + B = \begin{bmatrix} 1 + 2 & 2 + 1 & 3 + 5 & 4 - 1 \\ 2 + 8 & 4 + 6 & 6 + 6 & 8 + 2 \\ 3 + 3 & 6 - 1 & 9 + 2 & 12 - 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 8 & 3 \\ 10 & 10 & 12 & 10 \\ 6 & 5 & 11 & 9 \end{bmatrix} \]
Addition and Scalar Multiplication

Addition and scalar multiplication work the way you want them to.

\[ A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 5 & -1 \\ 8 & 6 & 6 & 2 \\ 3 & -1 & 2 & -3 \end{bmatrix} \]

\[ A + B = \begin{bmatrix} 1+2 & 2+1 & 3+5 & 4-1 \\ 2+8 & 4+6 & 6+6 & 8+2 \\ 3+3 & 6-1 & 9+2 & 12-3 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 8 & 3 \\ 10 & 10 & 12 & 10 \\ 6 & 5 & 11 & 9 \end{bmatrix} \]
Addition and Scalar Multiplication

Addition and scalar multiplication work the way you want them to.

\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 \\
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3 & 6 & 9 & 12
\end{bmatrix}, \quad B = \begin{bmatrix}
2 & 1 & 5 & -1 \\
8 & 6 & 6 & 2 \\
3 & -1 & 2 & -3
\end{bmatrix}
\]

\[
A + B = \begin{bmatrix}
1 + 2 & 2 + 1 & 3 + 5 & 4 - 1 \\
2 + 8 & 4 + 6 & 6 + 6 & 8 + 2 \\
3 + 3 & 6 - 1 & 9 + 2 & 12 - 3
\end{bmatrix} = \begin{bmatrix}
3 & 3 & 8 & 3 \\
10 & 10 & 12 & 10 \\
6 & 5 & 11 & 9
\end{bmatrix}
\]

\[
10A = \begin{bmatrix}
10 & 20 & 30 & 40 \\
20 & 40 & 60 & 80 \\
30 & 60 & 90 & 120
\end{bmatrix}
\]
Mobile money minimization: matrix multiplication motivation

You’re comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three services: texts, minutes talking, and GB of data.
Mobile money minimization: matrix multiplication motivation

You’re comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three services: texts, minutes talking, and GB of data. You want to know, for each person and plan, what the cost will be.
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Input: plans $\times$ services and people $\times$ services
Output: plans $\times$ people
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**Input:** plans $\times$ services and people $\times$ services

**Output:** plans $\times$ people

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You’re comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three services: texts, minutes talking, and GB of data. You want to know, for each person and plan, what the cost will be.

Input: plans × services and people × services
Output: plans × people

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<td>Plan 3</td>
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You’re comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three servies: texts, minutes talking, and GB of data. You want to know, for each person and plan, what the cost will be.

**Input:** plans×services and people×services  
**Output:** plans×people

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Input: plans $\times$ services and people $\times$ services

Output: plans $\times$ people

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**Input:** plans \( \times \) services and people \( \times \) services

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You’re comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three services: texts, minutes talking, and GB of data. You want to know, for each person and plan, what the cost will be. 

Input: plans × services and people × services
Output: plans × people

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You’re comparing cell phone plans. For some number of plans and for some number of people, you have information about costs and usage of three services: texts, minutes talking, and GB of data. You want to know, for each person and plan, what the cost will be.

**Input:** plans$\times$services and people$\times$services  
**Output:** plans$\times$people

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Matrix Multiplication

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6
\end{bmatrix} \cdot 
\begin{bmatrix}
1 & 0 \\
2 & 1 \\
0 & 3
\end{bmatrix} = 
\begin{bmatrix}
5 & 11 \\
10 & 22
\end{bmatrix}
\]

In the product, the entry in the \(i\)th row and \(j\)th column comes from dotting the \(i\)th row and \(j\)th column of the matrices being multiplied.

\[ [1, 2, 3] \cdot [1, 2, 0] = 5 \]
Matrix Multiplication

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
2 & 1 \\
0 & 3
\end{bmatrix} =
\begin{bmatrix}
5 & 11 \\
10 & 22
\end{bmatrix}
\]

In the product, the entry in the \(i\)th row and \(j\)th column comes from dotting the \(i\)th row and \(j\)th column of the matrices being multiplied.

\[
[1, 2, 3] \cdot [1, 2, 0] = 5 \\
[1, 2, 3] \cdot [0, 1, 3] = 11
\]
Matrix Multiplication

\[
\begin{pmatrix}
1 & 2 & 3 \\
2 & 4 & 6
\end{pmatrix}
\cdot
\begin{pmatrix}
1 & 0 \\
2 & 1 \\
0 & 3
\end{pmatrix}
= \begin{pmatrix}
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10 & 22
\end{pmatrix}
\]

In the product, the entry in the \(i\)th row and \(j\)th column comes from dotting the \(i\)th row and \(j\)th column of the matrices being multiplied.

\[
[1, 2, 3] \cdot [1, 2, 0] = 5 \\
[1, 2, 3] \cdot [0, 1, 3] = 11 \\
[2, 4, 6] \cdot [1, 2, 0] = 10
\]
Matrix Multiplication

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 \\
2 & 1 \\
0 & 3
\end{bmatrix}
= 
\begin{bmatrix}
5 & 11 \\
10 & 22
\end{bmatrix}
\]

In the product, the entry in the \(i\)th row and \(j\)th column comes from dotting the \(i\)th row and \(j\)th column of the matrices being multiplied.

\[
[1, 2, 3] \cdot [1, 2, 0] = 5 \\
[1, 2, 3] \cdot [0, 1, 3] = 11 \\
[2, 4, 6] \cdot [1, 2, 0] = 10 \\
[2, 4, 6] \cdot [0, 1, 3] = 22
\]
Matrix Multiplication

\[
\begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
2 & 1 \\
0 & 3
\end{bmatrix}
= 
\begin{bmatrix}
5 & 11 \\
10 & 22
\end{bmatrix}
\]

In the product, the entry in the ith row and jth column comes from dotting the ith row and jth column of the matrices being multiplied.

\[
\begin{align*}
[1, 2, 3] \cdot [1, 2, 0] &= 5 \\
[1, 2, 3] \cdot [0, 1, 3] &= 11 \\
[2, 4, 6] \cdot [1, 2, 0] &= 10 \\
[2, 4, 6] \cdot [0, 1, 3] &= 22
\end{align*}
\]
Another Example

\[
\begin{bmatrix}
0 & 1 & 3 \\
1 & 0 & 2 \\
1 & 1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
2 & 3 \\
3 & 0 \\
1 & 2 \\
\end{bmatrix}
= 
\begin{bmatrix}
6 & 6 \\
4 & 7 \\
6 & 5 \\
\end{bmatrix}
\]
Another Example

\[
\begin{bmatrix}
0 & 1 & 3 \\
1 & 0 & 2 \\
1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
2 & 3 \\
3 & 0 \\
1 & 2
\end{bmatrix}
= 
\begin{bmatrix}
6 & 6 \\
4 & 7 \\
6 & 5
\end{bmatrix}
\]
Another Example

\[
\begin{bmatrix}
2 & 5 \\
0 & 1 \\
1 & 1 \\
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & 3 & 1 \\
0 & 1 & 1 & 1 \\
\end{bmatrix}
= 
\begin{bmatrix}
\text{?} & \text{?} & \text{?} & \text{?} \\
\text{?} & \text{?} & \text{?} & \text{?} \\
\text{?} & \text{?} & \text{?} & \text{?} \\
\end{bmatrix}
\]
Another Example

\[
\begin{bmatrix} 2 & 5 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 11 & 7 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 4 & 2 \end{bmatrix}
\]
Wait but... why

\[
\begin{bmatrix}
x_1 + 2x_2 + 3x_3 + 4x_4 \\
5x_1 + 6x_2 + 7x_3 + 8x_4
\end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\]
Wait but... why

\[
\begin{bmatrix}
1x_1 + 2x_2 + 3x_3 + 4x_4 \\
5x_1 + 6x_2 + 7x_3 + 8x_4
\end{bmatrix} = \begin{bmatrix}
0 \\
2
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 0 \\
5 & 6 & 7 & 8 & 2
\end{bmatrix}
\]
Wait but... why

\[
\begin{bmatrix}
1x_1 + 2x_2 + 3x_3 + 4x_4 \\
5x_1 + 6x_2 + 7x_3 + 8x_4
\end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8
\end{bmatrix}
\]

\[
A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\]

\[Ax = b\]
We can only take the dot product of two vectors that have the same length.
We can only take the dot product of two vectors that have the same length.

If $A$ is an $m$-by-$n$ matrix, and $B$ is an $r$-by-$c$ matrix, then $AB$ is only defined if $n = r$. If $n = r$, then $AB$ is an $m$-by-$c$ matrix.
Dimensions

We can only take the dot product of two vectors that have the same length.

If $A$ is an $m$-by-$n$ matrix, and $B$ is an $r$-by-$c$ matrix, then $AB$ is only defined if $n = r$. If $n = r$, then $AB$ is an $m$-by-$c$ matrix.

Can you always multiply a matrix by itself?
Properties of Matrix Multiplication

One important property DOESN’T hold.

\[
\begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
7 & 5 \\
3 & 0
\end{bmatrix}
= \\
\begin{bmatrix}
13 & 5 \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
7 & 5 \\
3 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}
= \\
\begin{bmatrix}
7 & 14 \\
3 & 6
\end{bmatrix}
\]

Matrix multiplication is not commutative. Order matters.

Suppose the matrix product \(AB\) exists. Does the product \(BA\) also have to exist?
Properties of Matrix Multiplication

One important property DOESN’T hold.

\[
\begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
7 & 5 \\
3 & 0
\end{bmatrix}
= 
\begin{bmatrix}
13 & 5 \\
0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
7 & 5 \\
3 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
0 & 0
\end{bmatrix}
= 
\]

Matrix multiplication is not commutative.
Order matters.

Suppose the matrix product \( AB \) exists. Does the product \( BA \) also have to exist?
Properties of Matrix Multiplication

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13 & 5 \\
0 & 0 \\
\end{bmatrix}
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\begin{bmatrix}
1 & 2 \\
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= 
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7 & 14 \\
3 & 6 \\
\end{bmatrix}
\]

Matrix multiplication is not commutative. *Order matters.*
Properties of Matrix Multiplication

One important property DOESN’T hold.

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\begin{bmatrix}
1 & 2 \\
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\begin{bmatrix}
7 & 5 \\
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\begin{bmatrix}
13 & 5 \\
0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
7 & 5 \\
3 & 0 \\
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
0 & 0 \\
\end{bmatrix}
= 
\begin{bmatrix}
7 & 14 \\
3 & 6 \\
\end{bmatrix}
\]

Matrix multiplication is not commutative. *Order matters.*

Suppose the matrix product \( AB \) exists. Does the product \( BA \) also have to exist?
The other properties hold as you would like. (Page 128, notes.)

1. \( A + B = B + A \)
2. \( A + (B + C) = (A + B) + C \)
3. \( s(A + B) = sA + sB \)
4. \( (s + t)A = sA + tA \)
5. \( (st)A = s(tA) \)
6. \( 1A = A \)
7. \( A + 0 = A \) (where 0 is the matrix of all zeros)
8. \( A - A = A + (-1)A = 0 \)
9. \( A(B + C) = AB + AC \)
10. \( (A + B)C = AC + BC \)
11. \( A(BC) = (AB)C \)
12. \( s(AB) = (sA)B = A(sB) \)
Examples

Simplify the following expressions.

1) \[
\begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 3
\end{bmatrix}
\begin{bmatrix}
8 & 9 & 8 \\
9 & 8 & 9 \\
8 & 9 & 8
\end{bmatrix}
+ \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
1 & 2 & 3
\end{bmatrix}
\begin{bmatrix}
-8 & -9 & -8 \\
-9 & -8 & -9 \\
-8 & -9 & -8
\end{bmatrix}
\]

2) \[
\begin{bmatrix}
33 & 44 \\
55 & 66
\end{bmatrix}
\begin{bmatrix}
5 & 1 \\
7 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
1 & 1
\end{bmatrix}
\]

3) \[
2.8 \begin{bmatrix}
15 & 0 & 38 \\
9 & 10 & 11 \\
8 & 7 & 6
\end{bmatrix}
+ 5.6 \begin{bmatrix}
-2.5 & 0 & 1 \\
0.5 & 0 & -0.5 \\
1 & 1.5 & 2
\end{bmatrix}
\]
More on Dimensions

Suppose $A$ is an $m$-by-$n$ matrix, and $B$ is an $r$-by-$c$ matrix.

If we want to multiply $A$ and $B$, what has to be true about $m$, $n$, $r$, and $c$?

If we want to add $A$ and $B$, what has to be true about $m$, $n$, $r$, and $c$?

If we want to compute $(A + B)A$, what has to be true about $m$, $n$, $r$, and $c$?
Functions and Transformations

\[ f(v) = \|v\| \]

\[ f(u, v) = u \times v \]
Functions and Transformations

\[ f(v) = \|v\| \]
Functions and Transformations

\[ f(v) = \|v\| \]

\[ f : \text{vectors} \rightarrow \mathbb{R} \]
Functions and Transformations

\[ f(v) = 3v \]
Functions and Transformations

\[ f(v) = 3v \]
Functions and Transformations

\[ f(u, v) = u \times v \]
Functions and Transformations

The function $f$ maps pairs of vectors in $\mathbb{R}^3$ to vectors in $\mathbb{R}^3$ as follows:

\[ f(u, v) = u \times v \]
Linear Transformations

\[ f(x) = x^2 \]
Linear Transformations

\[ f(x) = x^2 \]

\[ f(2 + 3) = 25 \quad f(2) + f(3) = 4 + 9 = 13 \]
Linear Transformations

\[ f(x) = x^2 \]

\[ f(2 + 3) = 25 \]
\[ f(2 \times 3) = 36 \]

\[ f(2) + f(3) = 4 + 9 = 13 \]
\[ 2f(3) = 2 \times 9 = 18 \]
Linear Transformations

\[ f(x) = x^2 \]

\[ f(2 + 3) = 25 \quad f(2) + f(3) = 4 + 9 = 13 \]
\[ f(2 \times 3) = 36 \quad 2f(3) = 2 \cdot 9 = 18 \]

\[ g(x) = 5x \]

\[ g(2 + 3) = 25 \quad g(2) + g(3) = 10 + 15 = 25 \]
Linear Transformations

**f**

\[ f(x) = x^2 \]

\[
\begin{align*}
    f(2 + 3) &= 25 \\
    f(2 \times 3) &= 36 \\
    f(2) + f(3) &= 4 + 9 = 13 \\
    2f(3) &= 2 \cdot 9 = 18
\end{align*}
\]

**g**

\[ g(x) = 5x \]

\[
\begin{align*}
    g(2 + 3) &= 25 \\
    g(2 \times 3) &= 30 \\
    g(2) + g(3) &= 10 + 15 = 25 \\
    2g(3) &= 2 \cdot 15 = 30
\end{align*}
\]
Linear Transformations

\[ f(x) = x^2 \]

\[
\begin{align*}
  f(2 + 3) &= 25 \\
  f(2 \times 3) &= 36 \\
  f(2) + f(3) &= 4 + 9 = 13 \\
  2f(3) &= 2 \cdot 9 = 18
\end{align*}
\]

\[ g(x) = 5x \]

\[
\begin{align*}
  g(2 + 3) &= 25 \\
  g(2 \times 3) &= 30 \\
  g(2) + g(3) &= 10 + 15 = 25 \\
  2g(3) &= 2 \cdot 15 = 30 \\
  g(x + y) &= 5(x + y) = 5x + 5y = g(x) + g(y) \\
  g(xy) &= 5(xy) = x(5y) = xg(y)
\end{align*}
\]
Linear Transformations

Definition

A transformation $T$ is called **linear** if, for any $x, y$ in the domain of $T$, and any scalar $s$,

$$T(x + y) = T(x) + T(y)$$

and

$$T(sx) = sT(x).$$
Linear Transformations

Definition

A transformation $T$ is called **linear** if, for any $x, y$ in the domain of $T$, and any scalar $s$,

$$T(x + y) = T(x) + T(y)$$

and

$$T(sx) = sT(x).$$

Is differentiation $T(f(x)) = \frac{d}{dx}[f(x)]$ (of functions whose derivatives exist everywhere) a linear transformation?

Let $T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ 2x \end{bmatrix}$. Is $T$ a linear transformation?
Linear Transformations

Definition

A transformation $T$ is called \textbf{linear} if, for any $x, y$ in the domain of $T$, and any scalar $s$, 

$$T(x + y) = T(x) + T(y)$$

and 

$$T(sx) = sT(x).$$

Is every line $(T(x) = mx + b)$ a linear transformation?
Definition

A transformation $T$ is called **linear** if, for any $x, y$ in the domain of $T$, and any scalar $s$,

$$ T(x + y) = T(x) + T(y) $$

and

$$ T(sx) = sT(x). $$

Is every line ($T(x) = mx + b$) a linear transformation?

If $A$ is a matrix, then the transformation

$$ T(x) = Ax $$

of a vector $x$ is linear.
Example

Let $T(x)$ be the rotation of $x$ by ninety degrees.
Example

Let $T(x)$ be the rotation of $x$ by ninety degrees.
Example

Let $T(x)$ be the rotation of $x$ by ninety degrees.
Example

Let $T(x)$ be the rotation of $x$ by ninety degrees.
Example

Let \( T(x) \) be the rotation of \( x \) by ninety degrees.
Example

Let $T(x)$ be the rotation of $x$ by ninety degrees.
Example

Let \( T(x) \) be the rotation of \( x \) by ninety degrees.
Example

Let \( T(x) \) be the rotation of \( x \) by ninety degrees.

Rotation by a fixed angle is a linear transformation.
Computing Rotations

\[ T(v) = \|v\| \cos(\theta + \phi) \begin{bmatrix} \cos(\theta + \phi) \\ \sin(\theta + \phi) \end{bmatrix} = \begin{bmatrix} \cos(\theta + \phi) \\ \sin(\theta + \phi) \end{bmatrix} \]

\[ \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \]

\[ \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix} = \begin{bmatrix} \cos(\theta) \cos(\phi) - \sin(\theta) \sin(\phi) \\ \sin(\theta) \cos(\phi) + \cos(\theta) \sin(\phi) \end{bmatrix} = \begin{bmatrix} \cos(\theta + \phi) \\ \sin(\theta + \phi) \end{bmatrix} \]
Computing Rotations

\[
\begin{align*}
\|v\| \cos \theta &= \|v\| \cos (\theta + \phi) \\
\|v\| \sin \theta &= \|v\| \sin (\theta + \phi) \\
\cos \theta \cos \phi - \sin \theta \sin \phi &= \cos (\theta + \phi) \\
\sin \theta \cos \phi + \cos \theta \sin \phi &= \sin (\theta + \phi)
\end{align*}
\]
Computing Rotations

\[ \| \mathbf{v} \| \sin \theta \]

\[ \| \mathbf{v} \| \cos \theta \]
Computing Rotations

\[ T(v) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} v \]

\[ \|v\| \sin \theta \to \|v\| \sin (\theta + \phi) \]

\[ \|v\| \cos \theta \to \|v\| \cos (\theta + \phi) \]

\[ \cos \theta \cos \phi - \sin \theta \sin \phi \]

\[ \sin \theta \cos \phi + \cos \theta \sin \phi \]
Computing Rotations

\[ \|v\| \sin(\theta + \phi) \]

\[ T(v) \]

\[ \|v\| \sin \theta \]

\[ \|v\| \cos(\theta + \phi) \]

\[ \|v\| \cos \theta \]
Computing Rotations

\[ \|v\| \sin(\theta + \phi) \]

\[ T(v) \]

\[ \|v\| \sin \theta \]

\[ v \]

\[ \|v\| \cos(\theta + \phi) \]

\[ \|v\| \cos \theta \]

\[ \cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi \]

\[ \sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi \]
Computing Rotations

\[ \|v\| \sin(\theta + \phi) \]

\[ \|v\| \cos(\theta + \phi) \]

\[ v = [v_1, v_2]; \quad T(v) = [x, y] \]

\[ x = \|v\| \cos(\theta + \phi) \]
\[ = \|v\| (\cos \theta \cos \phi - \sin \phi \sin \theta) \]
\[ = v_1 \cos \phi - v_2 \sin \phi \]

\[ y = \|v\| \sin(\theta + \phi) \]
\[ = \|v\| (\sin \theta \cos \phi + \cos \theta \sin \phi) \]
\[ = v_1 \sin \phi + v_2 \cos \phi \]
Computing Rotations

\[ \mathbf{v} = [v_1, v_2]; \quad T(\mathbf{v}) = [x, y] \]

\[
x = \|\mathbf{v}\| \cos(\theta + \phi) = \|\mathbf{v}\| (\cos \theta \cos \phi - \sin \phi \sin \theta) = v_1 \cos \phi - v_2 \sin \phi
\]

\[
y = \|\mathbf{v}\| \sin(\theta + \phi) = \|\mathbf{v}\| (\sin \theta \cos \phi + \cos \theta \sin \phi) = v_1 \sin \phi + v_2 \cos \phi
\]
Computing Rotations

\( \mathbf{v} = [v_1, v_2] \); \hspace{1cm} T(\mathbf{v}) = [x, y]

\[
x = \|\mathbf{v}\| \cos(\theta + \phi) \\
= \|\mathbf{v}\| (\cos \theta \cos \phi - \sin \phi \sin \theta) \\
= v_1 \cos \phi - v_2 \sin \phi
\]

\[
y = \|\mathbf{v}\| \sin(\theta + \phi) \\
= \|\mathbf{v}\| (\sin \theta \cos \phi + \cos \theta \sin \phi) \\
= v_1 \sin \phi + v_2 \cos \phi
\]

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}
\]
Computing Rotations

\[ v = [v_1, v_2]; \quad T(v) = [x, y] \]

\[
\begin{align*}
  x &= \|v\| \cos(\theta + \phi) \\
  &= \|v\| (\cos \theta \cos \phi - \sin \phi \sin \theta) \\
  &= v_1 \cos \phi - v_2 \sin \phi \\
\end{align*}
\]

\[
\begin{align*}
  y &= \|v\| \sin(\theta + \phi) \\
  &= \|v\| (\sin \theta \cos \phi + \cos \theta \sin \phi) \\
  &= v_1 \sin \phi + v_2 \cos \phi \\
\end{align*}
\]

\[
\begin{bmatrix}
x \\
y
\end{bmatrix} = \begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix} \begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
\]

The matrix is called a rotation matrix, Rot_{\phi}
Computing Rotations

\[
\text{Rot}_\phi = \begin{bmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{bmatrix}
\]

What matrix should you multiply \[\begin{bmatrix} 4 \\ 2 \end{bmatrix}\] by to rotate it 90 degrees?
Computing Rotations

\[ \text{Rot}_\phi = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \]

What matrix should you multiply \[ \begin{bmatrix} 4 \\ 2 \end{bmatrix} \] by to rotate it 90 degrees?

\[ \text{Rot}_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]
Computing Rotations

\[ \text{Rot}_\phi = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \]

What matrix should you multiply \[ \begin{bmatrix} 4 \\ 2 \end{bmatrix} \] by to rotate it 90 degrees?

\[ \text{Rot}_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

What matrix should you multiply \[ \begin{bmatrix} 4 \\ 2 \end{bmatrix} \] by to rotate it 30 degrees?
Computing Rotations

\[ \text{Rot}_\phi = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \]

What matrix should you multiply \[ \begin{bmatrix} 4 \\ 2 \end{bmatrix} \] by to rotate it 90 degrees?

\[ \text{Rot}_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

What matrix should you multiply \[ \begin{bmatrix} 4 \\ 2 \end{bmatrix} \] by to rotate it 30 degrees?

\[ \text{Rot}_{\pi/6} = \begin{bmatrix} \sqrt{3}/2 & -\frac{1}{2} \\ \frac{1}{2} & \sqrt{3}/2 \end{bmatrix} \]
Computing Rotations

\[ \text{Rot}_\phi = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \]

What matrix should you multiply \[ \begin{bmatrix} 4 \\ 2 \end{bmatrix} \] by to rotate it 90 degrees?
\[ \text{Rot}_{\pi/2} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \]

What matrix should you multiply \[ \begin{bmatrix} 4 \\ 2 \end{bmatrix} \] by to rotate it 30 degrees?
\[ \text{Rot}_{\pi/6} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \]

Are rotations commutative?