Eigenvalues + Friends

B6: Consider the three-component differential equation $\mathbf{x}' = A\mathbf{x}$. The 3×3 matrix A has real entries. It has an eigenvalue $\lambda_1 = -2$ and an eigenvalue $\lambda_2 = -1 + i$ with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1+i \\ 1 \end{bmatrix}.$$

(a) [2 marks] Write the general solution to the differential equation.

(b) [2] Write the solution of the differential equation with initial data $\mathbf{x}(0) = [1, 2, 3]^T$.Your solution must be in real form, that is is cannot involve complex numbers or complex exponentials.

(c) [1] Describe all initial conditions for which the solution $\mathbf{x}(t)$ exhibits oscillatory behaviour. Justify your answer briefly.

A29: The matrix below represents rotation in 3D about a line through the origin.

$$\begin{bmatrix} 1/2 & -1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & 1/\sqrt{2} & 1/2 \end{bmatrix}$$

Find a vector in the direction of the line of rotation.

A30: A solution to the two component differential equation system $\mathbf{y}' = A\mathbf{y}$ is

$$\mathbf{y}(t) = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{2t}(\cos t + i \sin t).$$

The 2×2 matrix A has real entries. What is A?

A9: Find an eigenvector corresponding to eigenvalue 2 for the matrix below:

$$\begin{bmatrix} 5 & -3 \\ 6 & -4 \end{bmatrix}$$

A23: The matrix

$$\mathbf{A} = \begin{bmatrix} 7 & 0 & -10 \\ 5 & -3 & -5 \\ 5 & 0 & -8 \end{bmatrix}.$$

has the eigenvalue -3. Find all eigenvectors corresponding to this eigenvalue.

B4: Consider the differential equation system

$$\mathbf{x}' = \mathbf{A}\mathbf{x}.$$

where $\mathbf{A}$ has the eigenvalues $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$ with the corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1+i \\ 1-i \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 1-i \\ 1+i \end{bmatrix}.$$

(a) [2 marks] Write the general solution of the DE system (in either real or complex form).

(b) [2] Find the solution that satisfies $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in real form (no complex numbers or complex exponentials).

(c) [1] Write MATLAB commands that would compute the first column of the matrix $\mathbf{A}$ from the information given about the eigenvalues and eigenvectors of $\mathbf{A}$.

B3: Consider the 3×3 matrix

$$A = \begin{bmatrix} 2 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & -4 & 5 \end{bmatrix}.$$

(a) [1 mark] Find an eigenvector of A corresponding to the eigenvalue $\lambda = 1$.

(b) [2] Find all other eigenvalues of A.

(c) [2] Find a basis of eigenvectors of A.

B6: The matrix

$$\boldsymbol{A} = \begin{bmatrix} 1/2 & -\sqrt{2}/2 & 1/2 \\ \sqrt{2}/2 & 0 & -\sqrt{2}/2 \\ 1/2 & \sqrt{2}/2 & 1/2 \end{bmatrix}$$

represents a rotation in 3D relative to some axis.

(a) [2 marks] Find all the eigenvalues of $\boldsymbol{A}$.

(b) [1] Find the direction vector of the axis of the rotation. *Hint:* this vector remains unchanged after the rotation.

(c) [2] Find the angle of rotation around the axis in (b). *Hint:* rotate a vector that is perpendicular to the rotation axis.

A18: Write a matrix whose eigenvalues are the roots of the polynomial

$$z^4 + 3z^3 - 2z^2 + 4 = 0.$$

Do not try to find the roots. *Hint:* Remember your computer lab #6.

A23: A 2×2 matrix $\mathbf{A}$ is known to have eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Find the general solution to $\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}$.

A24: For the same system as in **A23** above, find the solution with initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

A18: Consider the differential equation system for $\mathbf{x}(t)$

$$\frac{d}{dt}\mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \mathbf{x}$$

Write the general solution to the system.

B1: Let $\mathbf{A}$ be a 2×2 matrix in the form

$$\begin{bmatrix} 2 & 4 \\ 1 & a \end{bmatrix}$$

(a) [1 mark] Find a such that $\mathbf{A}$ is not invertible.

(b) [1] For the value of a above, find the eigenvalues of $\mathbf{A}$.

(c) [2] Find the eigenvectors associated with the eigenvalues found above.

(d) [1] Do the eigenvectors found above form a basis of $\mathbb{R}^2$? Justify briefly.

A25: A 2×2 matrix $\mathbf{A}$ is known to have eigenvalue $\lambda_1 = 1+i$ with corresponding eigenvector

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Find the general solution to $\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}$.

A26: For the same system as in **A25** above, find the solution with initial conditions

$$\mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Your answer should be in real form (involving no complex quantities).

A7: Find an eigenvector of eigenvalue 2 for the matrix

$$\begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$$

A6: Let $\mathbf{A}$ be a 2×2 matrix with real entries. Suppose that

$$\begin{bmatrix} 1 \\ i \end{bmatrix}$$

is an eigenvector of $\mathbf{A}$ with eigenvalue $2 + 3i$. What is another eigenvalue of $\mathbf{A}$ and its associated eigenvector?

B2: Let $\mathbf{A}$ be the matrix

$$\begin{bmatrix} 4 & -1 & 7 \\ 0 & 3 & 0 \\ 1 & 2 & -2 \end{bmatrix}$$

(a) [2 marks] Find an eigenvector of $\mathbf{A}$ corresponding to eigenvalue $\lambda_1 = 3$.

(b) [2] Find all the other eigenvalues of $\mathbf{A}$.

(c) [1] How many linearly independent eigenvectors does $\mathbf{A}$ have? Justify briefly.

A28: Let $\mathbf{A}$ be a 2×2 matrix which represents a reflection across a line in $\mathbb{R}^2$. Suppose that $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is an eigenvector with eigenvalue 1 and $\begin{bmatrix} \alpha \\ 6 \end{bmatrix}$ is an eigenvector with eigenvalue $\beta \neq 1$. What are the values of α and β?

A27: Find the solution of the system of differential equations

$$\begin{aligned} x' &= 2x + y \\ y' &= x + 2y \end{aligned}$$

that satisfies the initial conditions $x(0) = 2$ and $y(0) = 3$.

A17: Consider

$$\mathbf{A} = \begin{bmatrix} 1 & y \\ 2 & z \end{bmatrix}.$$

Find y and z so that $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is an eigenvector of $\mathbf{A}$ with eigenvalue 2.

Complex Numbers

B5: Consider $z = -1/2 + i\sqrt{3}/2$. Recall that $\tan^{-1}\sqrt{3} = \pi/3$.

(a) [1] Mark the approximate location of z in a sketch of the complex plane.

(b) [1] Compute $|z|$.

(c) [1] Write z in polar form. That is, find a real number $r > 0$ and $0 \leq \theta < 2\pi$ such that $z = re^{i\theta}$.

(d) [2] Find real numbers a and b such that $(-1/2 + i\sqrt{3}/2)^{23} = a + bi$. Simplify your answer.

A19: Given that the real part x and imaginary part y of the complex number $z = x + iy$ satisfy the equation $(2 - i)x - (1 + 3i)y = 7$, find x and y.

A1: Compute

$$(2 + 3i)/(4 - 2i).$$

Your answer should be in the form $a + bi$ where a and b are real numbers to be determined.

A22: Let $z = 2 + 2i$. Write z^5 in the form $a + ib$ with a and b real numbers with no unevaluated trigonometric function values. *Hint:* first write z in polar form.

A27: What is the polar representation of $z = 1 + i$?

A28: What is the root of

$$z^5 = 3$$

that lies in the second quadrant of the complex plane? *Note:* the second quadrant has complex numbers with negative real parts and positive imaginary parts.

A28: Compute $\det(A)$, where A is the 3×3 matrix with complex entries given below. Your answer should be in the form $a + ib$.

$$A = \begin{bmatrix} 2+i & 3-i & 0 \\ 3+i & 2+i & 0 \\ 1 & 1 & i \end{bmatrix}$$

Linear Transformations

Questions **A10** and **A11** below concern the line L through the origin with direction $[3, -4]$.

A10: Find the projection of the vector

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

onto L.

A11: Find the matrix that represents reflection across the line L.

A19: Find the matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T(\mathbf{v}) = \mathbf{v} \times (1, 0, 0).$$

Here, $\times$ denotes the cross product.

A21: Consider the two perpendicular lines through the origin given below:

$$\begin{aligned} L_1 : &\quad x + 2y = 0 \\ L_2 : &\quad x - y/2 = 0 \end{aligned}$$

Find the matrix for the composition of linear transformations: projection onto L_1 followed by projection onto L_2.

A16: $\mathbf{A}$ is a 2×2 matrix such that

$$\mathbf{A}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \text{ and } \mathbf{A}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

Determine the entries of $\mathbf{A}$.

B4: Let T be a linear transformation. Suppose that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}.$$

(a) [1 mark] Calculate $T(\mathbf{e}_1)$, where $\mathbf{e}_1 = [1\ 0\ 0]^T$ is the unit vector in x_1-direction of $\mathbb{R}^3$.

(b) [2] Find the matrix representation of T.

(c) [2] Calculate $T^{20}(\mathbf{e}_1)$.

Random Walks

B5: The percentage of people with the disease, March Madness, is recorded every week. Note that it is possible to recover from March Madness one week and catch it again the following week. Records indicate that the disease can be modelled by a random walk and that if 50% of the population is infected with March Madness one week, then 60% of the population will be infected the next week. Records also indicate that if 100% of the population is infected one week, then 90% of the population will be infected the next week. It is known that 10% of the population has March Madness this week.

(a) [2 marks] What is the 2×2 probability transition matrix for this system?

(b) [1] What percentage of the population will have March Madness two weeks from now?

(c) [1] What percentage of the population had March Madness last week?

(d) [1] Approximately what will be the percentage of people with March Madness many weeks from now?

B3: Let

$$\mathbf{P} = \begin{bmatrix} 0.6 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.4 \\ 0.2 & 0.2 & 0.4 \end{bmatrix}$$

be the transition matrix of a random walk. It is known that $\mathbf{P}$ has three distinct eigenvalues. Two eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 0.2$ with corresponding eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \text{ and } \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

(a) [2 marks] Find the third eigenvalue λ_3 and its associated eigenvector $\mathbf{v}_3$.

(b) [2] If the initial state is

$$\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

find $\mathbf{x}_{10} = P^{10}\mathbf{x}_0$. *Hint:* $\mathbf{x}_0$ is a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$.

(c) [1] What is $\lim_{n\to\infty} x_n$?

A30: You are using MATLAB to investigate a large random walk (15 locations). You spend a long time entering the transition matrix P correctly. You looked at the eigenvalues of P and found they were all positive. You computed `A=P^3` and `B=P^4`. You then realized that you have over-written the matrix P but `A` and `B` are still saved. Describe briefly how you could recover P and not have to type it in all over again.

B4: Suppose in the year 2020, 50 million people live in cities and 50 million in the suburbs. Every year, 10% of city residents move to the suburbs and 20% of the residents of the suburbs move to cities.

(a) [1 mark] Write down the 2×2 probability transition matrix P for this problem, using the ordering (1) city and (2) suburbs.

(b) [1] What fraction of residents will be living in cities in 2022?

(c) [2] Find the eigenvalues of P and a basis of eigenvectors.

(d) [1] Assuming the overall population does not change (i.e., remains at 100 million), how many people will be living in the suburbs far in the future?

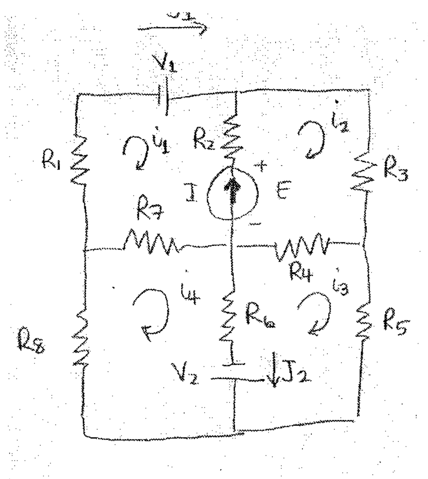
A30: Consider the probability transition matrix

$$\mathbf{P} = \begin{bmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/3 \\ 1/2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2/3 \end{bmatrix} \quad \text{and initial probability } \mathbf{x}_0 = \begin{bmatrix} 1/11 \\ 3/11 \\ 3/11 \\ 4/11 \end{bmatrix}$$

What is $\lim_{n\to\infty} \mathbf{P}^n \mathbf{x}_0$? *Hint:* this can be done without extensive calculations.

A4: Consider a random walk with three states. A walker that begins at state 1 always goes to state 2; always moves to state 3 when beginning in state 2; and is equally likely to be in any state after starting in state 3. Write the probability transition matrix for this random walk.

CIRCUITS



A20: Consider the resistor network above. Using the loop current method you have learned in the lectures and labs, write the equation for voltages around loop 1. Your equation should have the resistances and sources as parameters.

A21: Consider the resistor network above. Using the loop current method, write the equation that matches the loop currents to the current source.

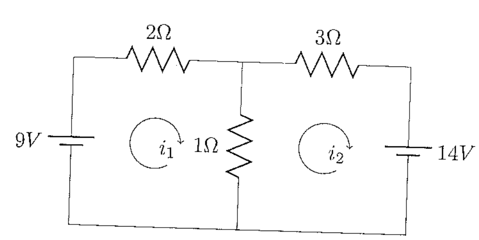
A22: Consider the resistor network above. Using MATLAB for a certain set of resistance values, it is found that

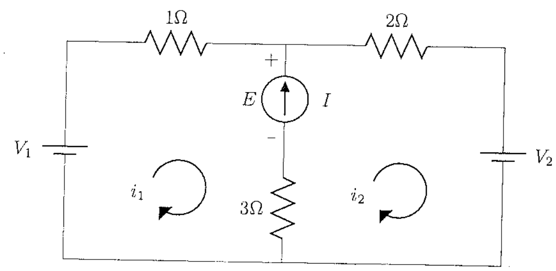
$$\begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ E \end{bmatrix} = \mathbf{A} \begin{bmatrix} V_1 \\ 0 \\ -V_2 \\ V_2 \\ I \end{bmatrix}, \quad \text{where} \quad \mathbf{A} = \begin{bmatrix} 7/27 & 7/27 & 10/81 & 4/81 & -43/81 \\ 7/27 & 7/27 & 10/81 & 4/81 & 38/81 \\ 4/27 & 4/27 & 25/81 & 10/81 & 14/81 \\ 1/9 & 1/9 & 4/27 & 7/27 & -1/27 \\ 13/27 & -14/27 & -20/81 & -8/81 & 248/81 \end{bmatrix}$$

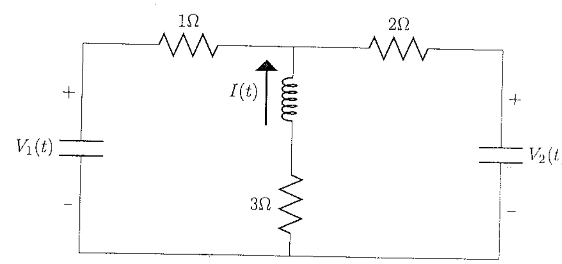
From this result, determine the *second row* of the matrix **B** such that

$$\begin{bmatrix} J_1 \\ J_2 \\ E \end{bmatrix} = \mathbf{B} \begin{bmatrix} V_1 \\ V_2 \\ I \end{bmatrix}$$

A11: Solve for the loop currents i_1 and i_2 in the circuit to the right.







Questions A25 and A26 concern the circuits above. In the left diagram, I is the current through the current source and E is the voltage across it, to be determined.

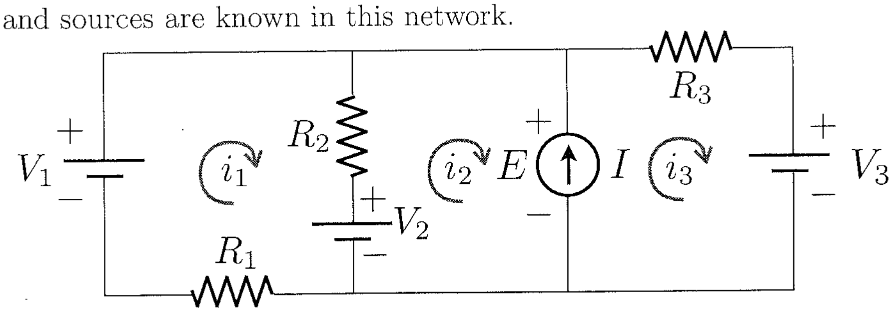
A25: For the left circuit above with two voltage sources and one current source write the *one* linear equation that matches the loop currents to the current source.

A26: The left circuit above has solution

$$\begin{aligned} i_1 &= -2I/3 + V_1/3 - V_2/3 \\ i_2 &= I/3 + V_1/3 - V_2/3 \\ E &= 11I/3 + 2V_1/3 + V_2/3 \end{aligned}$$

with all currents in Amps and potentials in Volts. Use this information to derive a differential equation system for $I(t)$, $V_1(t)$, and $V_2(t)$ in the right hand circuit where the two capacitors are 2 Farads and the inductor is 0.1 Henry.

For questions **A13-A15** below consider the resistor network in the diagram below. The resistances and sources are known in this network.



A13: List the unknowns in the linear system for this circuit using the technique of loop currents you learned in the lectures and computer labs this term.

A14: In terms of these unknowns, write the linear equation that represents Kirchhoff's law of voltage drops around the third loop (corresponding to i_3) in the circuit above.

A15: Write a linear equation that expresses the current through the current source in terms of the loop currents in the diagram.

MATLAB

A10: Consider the following lines of MATLAB code:

```
A = zeros(10,10);
for i=1:9
   A(i, i) = 1/2;
   A(i+1, i) = 1/2;
end
A(10,10) =1;
```

Circle the answer below that *best* describes the resulting matrix **A**:

(a) an error occurs in the last line above.

(b) **A** is the transition matrix for a random walk.

(c) **A** is an upper triangular matrix.

(d) **A** is a diagonal matrix with diagonal entries 1/2.

(e) **A** contains the solution of a differential equation system.

Problems **A14** and **A15** below concern solving systems of linear equations with MATLAB. For both questions, a system is entered into MATLAB as an augmented matrix `A`. In the two questions below, the result of `rref(A)` is shown. Use the information to determine whether each system has a solution or solutions. Either state that there is no solution and justify briefly or write down the solution or parametric form of all solutions.

A14:

```
1   0   3   0
0   1   0   0
0   0   0   1
```

A15:

```
1   0   1   2
0   1   1   1
```

A5: What is output after the following lines of MATLAB code?

```
A = [1 2 3 4 5; 6 7 8 9 10];
A(:,2)
```

A7: What is the result of the following MATLAB commands?

```
A = [1 2 3 4; 1 1 1 1; 9 8 7 6];
A(:,3)
```

A24: A 3×3 matrix `A` with real entries has been typed into MATLAB. The result of the command `[V D] = eig(A)` is (after some slight formatting changes to make it fit better in the exam):

```
V = 0.8165 + 0.0000i   0.8165 + 0.0000i   0.5774 + 0.0000i
    0.0000 + 0.0000i   0.0000 + 0.0000i   0.5774 + 0.0000i
    0.4082 - 0.4082i   0.4082 + 0.4082i   0.5774 + 0.0000i

D = 1.0000 + 2.0000i   0.0000 + 0.0000i   0.0000 + 0.0000i
    0.0000 + 0.0000i   1.0000 - 2.0000i   0.0000 + 0.0000i
    0.0000 + 0.0000i   0.0000 + 0.0000i  -1.0000 + 0.0000i
```

Circle *all* true statements below:

(a) `A` has no real eigenvalues.

(b) All eigenvalues of `A` have negative real parts.

(c) The eigenvectors of `A` are a basis for $\mathbb{R}^3$.

(d) Eigenvectors of `A` associated to distinct complex eigenvalues are linearly independent.

(e) $[1, 1, 1]^T$ is an eigenvector of `A`.

Geometric Applications

B5: Let P be the plane with equation $x_1 + x_2 + x_3 = 0$.

(a) [2 marks] Give a parametric form of the line L through an arbitrary point (a, b, c) which intersects P in a perpendicular direction.

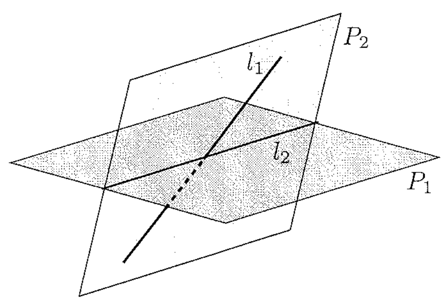
(b) [2] The projection onto the plane P is a linear transformation. Find its matrix representation $\mathbf{A}$. *Hint:* find the coordinates of the intersection of the line L above and P.

(c) [1] Explain briefly why 0 and 1 are eigenvalues of $\mathbf{A}$. Determine all eigenvectors of $\mathbf{A}$. Do both using a geometric argument.

A19: What is the shortest distance between points on the parallel lines?

$$\begin{aligned} 2x - y &= 2 \\ 4x - 2y &= 8 \end{aligned}$$

B3: Line l_1 and plane P_1 shown in the figure are given in parametric form as



$$l_1 : \boldsymbol{x} = t\boldsymbol{a}, \qquad P_1 : \boldsymbol{x} = \boldsymbol{q} + s_1\boldsymbol{b}_1 + s_2\boldsymbol{b}_2,$$

where

$$\boldsymbol{a} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad \boldsymbol{q} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{b}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

(a) [2 marks] Find the intersection point of the line l_1 and plane P_1.

(b) [2] Write the parametric form for plane P_2, which contains the line l_1 and is perpendicular to the plane P_1.

(c) [1] Planes P_1 and P_2 intersect along the line l_2. Write the equation form for the line l_2.

A2: Find the area of the triangle ABC in the plane, where $A = (3, 2)$, $B = (1, 3)$ and $C = (2, 5)$.

A14: Find the area of the parallelogram with vertices at $(2, -2)$, $(3, 1)$, $(5, 6)$, and $(4, 3)$.

A12: Find the distance from the point $(1, 2)$ to the line $x + y = 0$ in the plane.

A7: Let $\mathbf{a} = [1, 2, 3]$, $\mathbf{b} = [1, 1, 1]$ and $\mathbf{c} = [1, -4, 1]$. What is the volume of the parallelepiped generated by these vectors?

A26: Find an equation form of the line $[x, y, z] = [3 + 2s, s, 1 - 2s]$.

B1: Consider the lines

$$\begin{aligned} L_1 : &\quad [0, 2, 1] + s[-1, 2, 2] \\ L_2 : &\quad [-1, 0, 3] + t[-2, 1, 1] \end{aligned}$$

(a) [1 mark] Write two distinct points on L_1.

(b) [1] Write a vector that points in the direction parallel to L_1.

(c) [2] Do the lines L_1 and L_2 intersect? If so, find the intersection point. If not, explain.

(d) [1] Find a vector perpendicular to both L_1 and L_2.

A21: Let

$$L : \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 3 \\ a \end{bmatrix}$$

be a line in $\mathbb{R}^3$ that is parallel to $2x + 2y + z = 8$. Find a.

A15: Consider the line L passing through the point $P = [3, 2, 2]$ and which is perpendicular to the plane containing the points $A = [1, 0, 1]$, $B = [0, 1, 1]$, and $C = [-1, 0, 1]$. Give a parametric equation for L.

Points $A = [2, 1]$, $B = [5, 7]$, $C = [0, 4]$ and D form the vertices of a parallelogram $ABCD$ that is the subject of questions A4 and A5 below. *Note:* that B and D are points adjacent to A.

A4: What is the point D?

A5: What is the area of the parallelogram $ABCD$?

A11: Find the parametric form of the plane in $\mathbb{R}^3$ with equation form

$$2x + 3y + 4z = 7$$

A22: Let

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}.$$

Compute ABA^T.

A8: Find all solutions (x, y, z) to the linear system

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 5 \\ 2x & + & 2y & & & = & 6 \\ & & 2y & + & 4z & = & 8 \end{array}.$$

A8: Let $\mathbf{a} = [1, 2, 3]$, $\mathbf{b} = [1, 1, 1]$ and $\mathbf{c} = [1, -4, 1]$. Find the projection of $\mathbf{a}$ onto $\mathbf{b} + \mathbf{c}$.

A18: Find a constant a so that the following set of vectors is linearly *dependent*:

$$\{[a, 0, 1], [1, 2, 1], [4, 1, 3]\}.$$

A20: If A is a matrix with 5 rows and 4 columns such that the set of solutions to the homogeneous system $A\mathbf{x} = 0$ has 2 parameters, what is the rank of A?

A24: Write $\mathbf{b} = [-5, 11, 18]$ as a linear combination $\mathbf{a}_1 = [1, 2, 3]$ and $\mathbf{a}_2 = [3, -1, -2]$ or show it cannot be done.

A1: Calculate the projection of the vector $[3, 1, 5]$ onto the vector $[2, 2, 4]$.

A4: Let A be a 5×5 matrix with $\det(A) = 10$. What is $\det(A^{-1})$?

Let $z = 1 + 3i$ and $u = 1 - i$ be the complex numbers in questions A1-A3 below. Your answers should be in the form $a + ib$ where a and b are real numbers.

A1: Compute $z + 2u$.

A2: Compute zu.

A3: Compute z/u.

Questions A5-A6 below involve the vectors

$$\mathbf{u} = [1, 1, a] \quad \text{and} \quad \mathbf{v} = [1, 2, 3]$$

For each question A5-A6 below justify your answer with a short computation or a short justification in words. Note that the vector $\mathbf{u}$ has a constant a in the last component.

A5: For what value or values of a (if any) are $\mathbf{u}$ and $\mathbf{v}$ perpendicular?

A6: For what value or values of a (if any) are $\mathbf{u}$ and $\mathbf{v}$ parallel?

Matrix Algebra, Linear Systems

For questions **A16-A17** below, consider the system with parameters a and b:

$$\begin{aligned} 2x + y &= a \\ bx - 2y &= 6 \end{aligned}$$

A16: For what values of a and b (if any) does the system have a unique solution?

A17: For what values of a and b (if any) does the system have infinitely many solutions?

A13: Compute the determinant of **AB** where

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ 9 & 1 & 1 \end{bmatrix}$$

A12: Compute the determinant of

$$\begin{bmatrix} 3 & 1 & -1 & 1 \\ 1 & 0 & 2 & 3 \\ -2 & 0 & 3 & 4 \\ 1 & 0 & 2 & 2 \end{bmatrix}$$

A23: Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

A9: Compute the inverse of

$$\begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$$

A29: Find the inverse of

$$\begin{bmatrix} 0 & 2 & 1 \\ 2 & 4 & 5 \\ 1 & 2 & 1 \end{bmatrix}$$

A2: For what value or values of a (if any) is the matrix below invertible?

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & a & 2 \end{bmatrix}$$

A27: The set of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ can be written in parametric form

$$\mathbf{x} = [0, 1, 0, 1]t + [7, 0, 1, 0]s.$$

If A is a 3×4 matrix, what is the reduced row echelon form of A?

B2: In the system below, x, y, and z are variables, and a and b are constants.

$$\begin{array}{ccccccc} x & + & y & + & z & = & 5 \\ x & & & + & z & = & 1 \\ ax & & & + & z & = & b \end{array}$$

(a) [1 mark] Write the system as an augmented matrix.

(b) [2] Bring the augmented matrix to row echelon form.

(c) [1] For what value or values of a and b (if any) does the system have no solutions?

(d) [1] For what value or values of a and b (if any) does the system have an infinite number of solutions?

B1: Uno, Duo and Traea are three friends. They all owe money to a loan shark. All together they owe \$600. Duo owes \$200 more than Uno. Uno and Duo combined owe as much as Traea.

(a) [2 marks] Let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

be the vector of unknowns, where x_1, x_2 and x_3 is the amount of money Uno, Duo and Traea owe, respectively. Describe the information above as a linear system in the form

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

(write $\mathbf{A}$ and $\mathbf{b}$ with specific values).

(b) [1] Write the system you found above in augmented matrix form.

(c) [2] Solve the system above using Gaussian elimination on the augmented matrix. How much does each person owe?

B2: Let d be a constant to be determined. It is known that the linear system in augmented matrix form

$$\left[\begin{array}{ccccc|c} 1 & 2 & -1 & 0 & 4 & 1 \\ 2 & 3 & -1 & -1 & 5 & 0 \\ 0 & 1 & -1 & 1 & 3 & 2 \\ 3 & 5 & -2 & 0 & 10 & d \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 1 & 3 & 3 \\ 0 & 1 & -1 & -3 & -1 & -6 \\ 0 & 0 & 0 & 2 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

where $\sim$ denotes allowable operations to put the augmented matrix into row echelon form.

(a) [2 marks] Find the reduced row echelon form of the given system.

(b) [2] Find the solution or solutions of the system (if any).

(c) [1] What is the value of d?

A25: Find the value for b, for which $\boldsymbol{AB} = \boldsymbol{BA}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} b & 1 \\ 0 & 1 \end{bmatrix}.$$

A16: Consider the matrix representation A of a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^3$. Circle *all* correct answers below:

(a) A is invertible.

(b) A has three rows.

(c) A has three columns.

(d) $T(\mathbf{x}) = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$.

(e) $A = A^T$.

A17: Consider a linear system with 7 equations for 8 unknowns. Circle *all* possible types of solution sets that could result:

(a) The system has no solutions.

(b) The system has a unique solution.

(c) The system has exactly 8 distinct solutions.

(d) The system has a one-parameter family of solutions.

(e) The system has a two-parameter family of solutions.

A3: Circle the *one* correct answer below. A linear system of three equations in five unknowns has

(a) always a unique solution.

(b) either a unique solution or no solutions.

(c) either a unique solution or an infinite number of solutions.

(d) either no solutions or an infinite number of solutions.

(e) either a unique solution or exactly two distinct solutions.

For questions A9 and A10 below, consider the homogeneous system of equations represented by this augmented matrix in reduced row echelon form:

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

A9: What is the rank of the augmented matrix above?

A10: Write a parametric form for all solutions to the system above.

A20: Find the determinant of

← Find the determinant of

$$\begin{bmatrix} 1 & 3 & 2 & 0 & 1 \\ 2 & 7 & 1 & 0 & 2 \\ 1 & 5 & 6 & -1 & 7 \\ 0 & 0 & 3 & 0 & 0 \\ 1 & 1 & 4 & 0 & 4 \end{bmatrix}$$

B6: Let $\mathbf{v_1} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} 1 \\ 0 \\ a \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ b \end{bmatrix}$.

(a) [1 mark] For what value of a are the three vectors $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$ listed above linearly dependent?

(b) [2] For the value of a found in (a), find a linear relation between the three vectors, that is find scalars α, β, γ such that $\alpha\mathbf{v_1} + \beta\mathbf{v_2} + \gamma\mathbf{v_3} = \mathbf{0}$. Express the result as a vector $\mathbf{k} = [\alpha\ \beta\ \gamma]^T$.

(c) [1] For the value of a found in (a), let $A = [\mathbf{v_1}\ \mathbf{v_2}\ \mathbf{v_3}]$ and $\mathbf{x} = [x_1\ x_2\ x_3]^T$. For what value of b does $A\mathbf{x} = \mathbf{b}$ have a solution?

(d) [1] For the values of a, b found in (a) and (c), solve $A\mathbf{x} = \mathbf{b}$. Explain the relation between this solution and the result in (b).

A8: Circle *all* the statements below that *must be true* for any 3×3 matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$. It is known that $\det \mathbf{A} = 4$, $\det \mathbf{B} = 3$ and $\det \mathbf{C} = 0$.

(a) $\det(2\mathbf{A}) = 8$

(b) 0 is an eigenvalue of $\mathbf{C}$.

(c) $\det(\mathbf{A} + \mathbf{B}) = 7$.

(d) $\mathbf{ABC}$ is invertible.

(e) $\det \mathbf{B}^T = 3$.

A12: Circle *all* possible solution sets for linear systems of five equations in three unknowns.

(a) a unique solution.

(b) no solutions.

(c) an infinite number of solutions.

(d) exactly two solutions.

A13: Calculate the determinant of this matrix:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 10 & 0 & 3 & 10 \\ 7 & 1 & 5 & 9 \\ -1 & 0 & 0 & 1 \end{bmatrix}.$$

A6: Consider the following linear system written in augmented matrix form:

$$\left[\begin{array}{ccc|c} 0 & 1 & 2 & 5 \\ 0 & 2 & 1 & 7 \end{array}\right]$$

Write the solution or solutions, or state that none exist.

A29: Solve the matrix equation $3\mathbf{A} + \frac{1}{2}\mathbf{B}\mathbf{X} = \mathbf{C}$ for $\mathbf{X}$, in which $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{X}$ are all 2×2 matrices and that

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} -1 & 3 \\ 2 & -11 \end{bmatrix}$$