Things to Memorize:

A *Partial* List

January 27, 2017

Chapter 2

Vectors - Basic Facts

- A **vector** has a magnitude (also called size/length/norm) and a direction. It does not have a *fixed* position, so the same vector can be "moved around" as long as it doesn't grow/shrink or get tilted.
- If you multiply a vector by a positive scalar, the direction stays the same, and the length is scaled. If you multiply a vector by a negative scalar, the direction becomes opposite (but still parallel), and the length is scaled.
- We call two vectors **parallel** if they are scalar multiples of each other.
- We geometrically add two vectors together by placing the head of one at the tail of the other, and drawing the sum vector from the tail of the first to the head of the second. This is equivalent to taking a particular diagonal of the parallelogram formed by the two vectors.
- Using the above definition, we see that the *difference* of two vectors can be drawn geometrically by taking the two vectors with their tails together, and drawing a vector between their heads.
- $\mathbb{R}^n$ refers to the space of vectors with n coordinates. So, $\mathbb{R}^2$ is all vectors with two coordinates–that is, all points in the xy-plane. Similarly, $\mathbb{R}^3$ is all vectors with three coordinates–that is, all points in xyz-space.
- A **unit vector** is a vector with length 1.
- In $\mathbb{R}^2$, $\mathbf{i} = [1, 0]$ and $\mathbf{j} = [1, 0]$. So, they are the unit vectors in the directions of the positive axes. Similarly, in $\mathbb{R}^3$, $\mathbf{i} = [1, 0, 0]$, $\mathbf{j} = [0, 1, 0]$, and $\mathbf{k} = [0, 0, 1]$.
- The length of vector $\mathbf{a} = [a_1, a_2, \ldots, a_n]$ is

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + \cdots a_n^2}$$

Dot Product

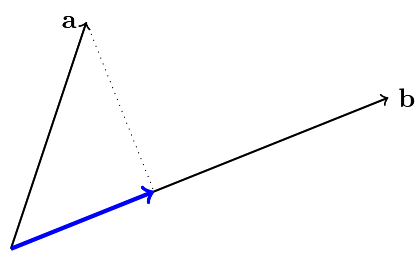
- The dot product combines two vectors (with the same number of coordinates) and gives a number. (By contrast, the cross product combines two vectors in $\mathbb{R}^3$ and gives a third vector in $\mathbb{R}^3$.)
- To calculate the dot product of two vectors, you multiply together their corresponding coordinates, and add the products. For example:

$$\begin{bmatrix} 8 \\ 3 \\ 10 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 5 \\ 6 \end{bmatrix} = \big[(8)(-1) + (3)(2) + (10)(5) + (3)(6) = 66\big]$$

- Given two vectors $\mathbf{a}$ and $\mathbf{b}$, when we take their dot product, it tells us something about the angle θ between the two of them:
$$\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$$
- It follows from the previous bullet point that two non-zero vectors are **orthogonal** (or **perpendicular**) if and only if their dot product is zero. This is a very common use of the dot product–determining orthogonality.
- $\|\mathbf{a}\|^2 = \mathbf{a}\cdot\mathbf{a}$

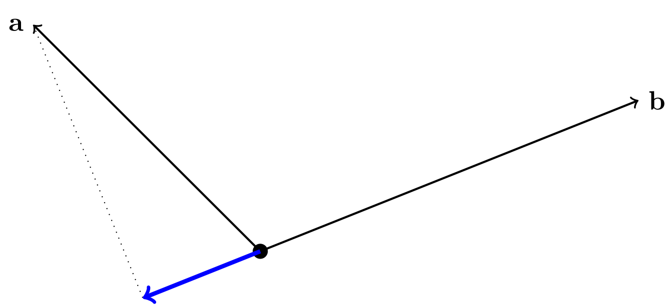
Projections

- The projection of $\mathbf{a}$ onto $\mathbf{b}$ is the component of $\mathbf{a}$ that is parallel to $\mathbf{b}$. The picture is this:



We draw a line from the tip of $\mathbf{a}$ to $\mathbf{b}$ that meets $\mathbf{b}$ at a right angle. The point where it meets $\mathbf{b}$ is the head of $\text{proj}_{\mathbf{b}}\mathbf{a}$, and the tail is the tail of $\mathbf{b}$. So, the blue vector is the projection.

We usually show these as acute angles, but they work for obtuse also. We really only care about the line in the direction of $\mathbf{b}$. If the angle is obtuse, the picture is like this:



- We can calculate a projection as follows:
$$\text{proj}_{\mathbf{b}}\mathbf{a} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{b}^2\|}\right)\mathbf{b}$$
- The dot product behaves algebraically together with scalar multiplication and vector addition much the way you would like it to. For example, $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) = \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}$.

Determinants of Small Matrices

- The determinant of a square matrix is a number. So the determinant function has a square matrix as its input, and a number as its output.

- To calculate the determinant of a 2-by-2 matrix:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

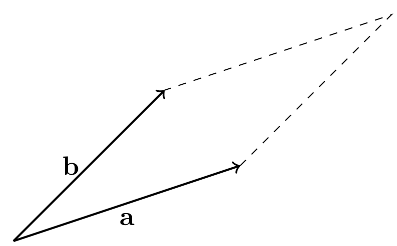
- If two non-zero vectors in $\mathbb{R}^2$ are parallel (that is, scalar multiples of each other) then when we put them as rows in a matrix, the determinant of that matrix will be 0.

- To calculate the determinant of a 3-by-3 matrix:

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

  That is: first, we take the top row, and alternate signs $+$, $-$, $+$. Then, to decide which sub-matrix we'll take the determinant of, we delete the row and column of the entry we chose.

- If $\mathbf{a}$ and $\mathbf{b}$ are two vectors in $\mathbb{R}^2$, then we can calculate the area of the parallelogram spanned by them (as in the picture below) by making them the rows of a 2-by-2 matrix and taking the absolute value of the determinant.



$$\text{Area of parallelogram} = \left| \det \begin{bmatrix} -\mathbf{a}- \\ -\mathbf{b}- \end{bmatrix} \right|$$

- Equivalently, the absolute value of the determinant is $\|\mathbf{a}\|\|\mathbf{b}\||\sin\theta|$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

Cross Product

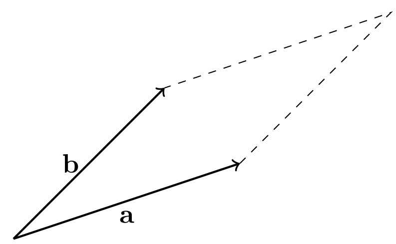
- The cross product combines two vectors in $\mathbb{R}^3$ to give a third vector in $\mathbb{R}^3$.

- Unlike dot products and determinants, we won't define a cross product for vectors except those with three coordinates.

- $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$; other properties of the cross product are given on Page 31 of your text

- $\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ - & \mathbf{a} & - \\ - & \mathbf{b} & - \end{bmatrix}$

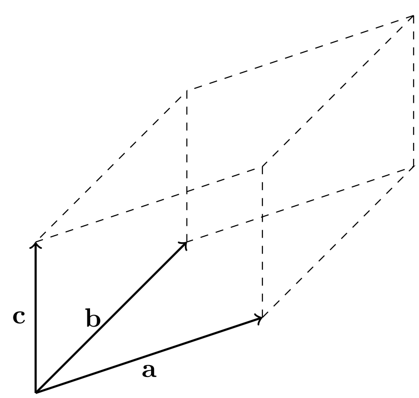
- $\|\mathbf{a} \times \mathbf{b}\|$ gives the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. That is, $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\||\sin\theta|$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. So, the *length* of the cross product is the area of the parallelogram spanned by the two vectors.



Area of parallelogram = $\|\mathbf{a} \times \mathbf{b}\|$

Note: contrast this to the area of the parallelogram formed by two vectors in $\mathbb{R}^2$.

- The vector $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$. However, we need to be more precise than this, because there are two directions that are orthogonal two both vectors. Its direction (up or down) is given by the right-hand rule.
- If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are three vectors in $\mathbb{R}^3$, then we can calculate the volume of the parallelepiped spanned by them by making them the rows of a 3-by-3 matrix and taking the absolute value of the determinant.



$$\text{Volume of parallelepiped} = \left| \det \begin{bmatrix} -\mathbf{a}- \\ -\mathbf{b}- \\ -\mathbf{c}- \end{bmatrix} \right|$$

- The determinant above is also the **triple product** of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, which can also be calculated as $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ or $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

Lines and Planes

- The parametric equation of a line (in any dimension: $\mathbb{R}^2$, $\mathbb{R}^3$, etc) is

$$\mathbf{x} = \mathbf{q} + s\mathbf{a}$$

  That is, the line consists of all points $\mathbf{x}$ that can be written as some scalar multiple of $\mathbf{a}$ plus the constant vector $\mathbf{q}$. The vector $\mathbf{a}$ gives the **direction** of the line, and the vector $\mathbf{q}$ is a point on the line.

- The component equation of a line in $\mathbb{R}^2$ is $ax + by = c$, where a, b, and c are constant scalars. The vector $[a, b]$ is orthogonal to the line.

- The parametric equation of a plane (in any dimension: $\mathbb{R}^3$, $\mathbb{R}^4$, etc) is

$$\mathbf{x} = \mathbf{q} + s\mathbf{a} + t\mathbf{b}$$

where $\mathbf{a}$ and $\mathbf{b}$ are not parallel. That is, the line consists of all points $\mathbf{x}$ that can be written as a linear combination of $\mathbf{a}$ and $\mathbf{b}$ plus the constant vector $\mathbf{q}$. The vectors $\mathbf{a}$ and $\mathbf{b}$ lie in the plane; that is, if we position them to their tail is a point on the plane, then their head is a point on the plane as well. (We also call a vector that lies in the plane "parallel" to the plane.) The constant vector $\mathbf{q}$ is a point on the plane.

- The component equation of a plane in $\mathbb{R}^3$ is $ax + by + cz = d$, where the vector $[a, b, c]$ is normal to the plane. That is, any vector parallel to the plane is orthogonal to $[a, b, c]$. If $d = 0$, then the plane passes through the origin. The constant d will determine the position of the plane in space, while the coefficients a, b, c, determine the plane's "tilt." Note that for every point (x, y, z) in the plane, $[x, y, z] \cdot [a, b, c] = d$.

- The component equation of a line in $\mathbb{R}^3$ is as the intersection of two planes:

$$\begin{cases} a_1x + a_2y + a_3z &= a_4 \\ b_1z + b_2y + b_3z &= b_4 \end{cases}$$

The vectors $[a_1, a_2, a_3]$ and $[b_1, b_2, b_3]$ are orthogonal to the line. To get a direction vector of the line, you can calculate the cross product of these two vectors.

Linear Systems

- Let $a_1x + a_2y = a_3$ and $b_1x + b_2y = b_3$ be lines that are not parallel. Then their intersection is a point. So, the solution to the following system of linear equations is a single point.

$$\begin{cases} a_1x + a_2y &= a_3 \\ b_1z + b_2y &= b_3 \end{cases}$$

- Let $a_1x + a_2y + b_3z = a_4$, $b_1x + b_2y + b_3z = b_4$, and $c_2x + c_2y + c_3z = c_4$ be planes in $\mathbb{R}^3$ whose normal vectors do not all lie on the same plane. (That is, their normal vectors are linearly independent.) Then their intersection is a point. So, the solution to the following system of linear equations is a single point.

$$\begin{cases} a_1x + a_2y + a_3z &= a_4 \\ b_1z + b_2y + b_3z &= b_4 \\ c_1z + c_2y + c_3z &= c_4 \end{cases}$$

- If three vectors lie on the same plane, then the parallelepiped they define is smooshed all the way flat, and so has volume zero, and hence determinant zero. So, the system of linear equations in the bullet point above has precisely one solution if and only if

$$\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \neq 0$$

In the case that the determinant is zero, the solutions to the system of linear equations could form a line (that is, they can be described using one parameter), they could form a plane (that is, it takes two parameters to describe them), they could be all of $\mathbb{R}^3$ (in the silly case that every equation is $0x + 0y + 0z = 0$), or there could be no solutions at all.

Linear Independence

- A **linear combination** of vectors is a vector formed by adding scalar multiples of the given vectors.

  For example, the vector $2\mathbf{a} - 5\mathbf{b} + \pi\mathbf{c}$ is a linear combination of the vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$.

- The set of all linear combinations of a collection of vectors is called the **span** of the vectors.

- A collection of vectors is **linearly independent** if the only linear combination of them to equal the zero vector is the linear combination where all scalars are zero.

  That is: consider a collection of vectors $\mathbf{a_1}$, $\mathbf{a_s}$, ..., $\mathbf{a_n}$. Set up the system of linear equations $s_1\mathbf{a_1} + s_2\mathbf{a_2} + \cdots + s_n\mathbf{a_n} = \mathbf{0}$. If the only solution is $s_1 = s_2 = \cdots = s_n = 0$, then the vectors are linearly independent. If there are other solutions, then the vectors are linearly dependent.

- The above is the formal definition of linear independence. For a collection of at least two vectors, it is equivalent to the following definition: a collection of vectors is linearly independent if no vector in the collection can be written as a linear combination of the others.

- A collection of n linearly independent vectors from $\mathbb{R}^n$ is called a **basis** of $\mathbb{R}^n$.

- Suppose you have a collection of vectors that form a basis of $\mathbb{R}^n$. Then any vector in $\mathbb{R}^n$ can be written as a linear combination of your vectors, and there is a unique way to do it.

Chapter 3

Solving Linear Systems

- You should know how to solve a linear system using substitution.

- We can use three "row operations" on a linear system without changing its solutions:

  1. Multiplying an equation (row) by a non-zero scalar
  2. Adding a multiple of one row to another row
  3. Swapping the vertical order of rows

- To save time and space, we use a shorthand for systems of linear equations. We use an **augmented matrix**, by which we mean we take a matrix (representing the coefficients of the variables) and augment it by adding an extra column separated by a vertical line (the column represents the constants in the equations).

- You should know how to use Gaussian Elimination (pivoting) to solve a system of linear equations.

- You should recognize when a matrix is in row echelon form and reduced row echelon form. (See the pictures on page 79 in the text.)

- The **rank** of a matrix (without its augmentation) is the number of nonzero rows that it has *after* being reduced.

- You should know how to express the solutions to a system of linear equations using parameters.

- The number of parameters needed to describe the solutions to a system of linear equations with rank r and n variables is $n - r$, *provided* there are any solutions at all.

- If a matrix has n rows and columns, and rank n, then its reduced row echelon form will have 1s along the main diagonal and 0s everywhere else.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  A system of linear equations that looks like this will have precisely one solution, regardless of what its constants are.

- If a matrix has n columns, rank n, and at least n rows, then its reduced row echelon form will have 1s along the main diagonal and 0s everywhere else.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

  A system of linear equations that looks like this will have either precisely one solution or no solutions at all.

Homogeneous Systems

- A system of linear equations is **homogeneous** if the constants are all zero. For instance, the system on the left is homogeneous, while the system on the right is not.

$$\left\{ \begin{array}{rcl} a_1x + a_2y & = & 0 \\ b_1z + b_2y & = & 0 \end{array} \right. \qquad \left\{ \begin{array}{rcl} a_1x + a_2y & = & 0 \\ b_1z + b_2y & = & 1 \end{array} \right.$$

- A homogeneous system always has at least one solution (where $x = y = z = \cdots = 0$)

- So, the solutions of a linear equation can always be written as $s_1\mathbf{a_1} + s_2\mathbf{a_2} + \cdots + s_k\mathbf{a_k}$ (for some appropriate k, and possibly the vectors are all $\mathbf{0}$).

- Any linear combination of solutions of A_0 is itself a solution of A_0.

- If you have a system of linear equations, you can form the **associated homogeneous system** by simply changing all the constants to 0. *This is a different system, with different solutions,* but the solutions are related in a predictable manner.

- Suppose A is a system of linear equations, and A_0 is its associated homogeneous system. Let $s_1\mathbf{a_1} + s_2\mathbf{a_2} + \cdots + s_k\mathbf{a_k}$ be the complete set of solutions to A_0. It's possible that A has no solutions. If it has at least one solution $\mathbf{q}$, then the complete set of solutions to A can be written as $\mathbf{q} + s_1\mathbf{a_1} + s_2\mathbf{a_2} + \cdots + s_k\mathbf{a_k}$. There are lots of related facts: for instance, the difference of any two solutions of A is a solution of A_0, and the sum of any solution of A with any solution of A_0 is a solution to A.