① Find eigenvalues

$$\det(A-\lambda I)=\det\begin{array}{cc} \begin{matrix}(+) & (-) & (+)\end{matrix} & \\ \begin{pmatrix}1-\lambda & 1 & 0\\ -1 & 1-\lambda & 0\\ 0 & 0 & -\lambda\end{pmatrix} & \begin{matrix} \\ (-) \\ (+)\end{matrix}\end{array}$$

$$-\lambda \det\begin{pmatrix}1-\lambda & 1\\ -1 & 1-\lambda\end{pmatrix} = \underbrace{-\lambda}\,\underbrace{\left[(1-\lambda)^2+1\right]} = 0$$

$\boxed{\lambda=0}$ or

$$(1-\lambda)^2+1=0$$

$$(1-\lambda)^2=-1$$

$$1-\lambda=\pm\sqrt{-1}=\pm i$$

$$\boxed{1\pm i=\lambda}$$

② Find eigenvectors

$\boxed{\lambda = 0}$

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{x_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}$$

$$\left.\begin{aligned} x + y &= 0 \\ -x + y &= 0 \end{aligned}\right\} \quad 2y = 0, \text{ so } \boxed{y = 0}$$

$$x + y = 0 \rightarrow \boxed{x = 0}$$

$$0 = 0 \qquad z : \text{anything}$$

$\boxed{\lambda = 1+i}$

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = (1+i)\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\cancel{x} + y = \cancel{x} + ix \rightarrow y = ix$$

$$-x + \cancel{y} = \cancel{y} + iy \rightarrow -x = iy$$

equiv (mult 1st by i)

$$0 = (1+i)z \rightarrow \boxed{z = 0}$$

eg $$\boxed{x_2 = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}}$$

Ignore $\lambda = 1 - i$; $x_3 = \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}$

General Solution:

$$\underbrace{c_1 e^{\lambda_1 t} x_1}_{\lambda_1 = 0} + \underbrace{c_2 \boxed{e^{\lambda_2 t} x_2} + c_3 e^{\lambda_3 t} x_3}_{\text{Instead of } \uparrow \text{, use:}}$$

$$c_2 \, Re(e^{\lambda_2 t} x_2) + c_3 \, Im(e^{\lambda_2 t} x_2)$$

$$e^{\lambda_2 t} x_2 = e^{(1+i)t} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} = e^t \underline{e^{it}} \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$$

$$= e^t (\underline{\cos t + i \sin t}) \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^t \cos t + i e^t \sin t \\ i e^t \cos t - e^t \sin t \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} e^t \cos t \\ -e^t \sin t \\ 0 \end{bmatrix}}_{Re} + i \underbrace{\begin{bmatrix} e^t \sin t \\ e^t \cos t \\ 0 \end{bmatrix}}_{Im}$$

General Solution:

$$c_1 e^{0t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} e^t \cos t \\ -e^t \sin t \\ 0 \end{bmatrix} + c_3\begin{bmatrix} e^t \sin t \\ e^t \cos t \\ 0 \end{bmatrix}$$

$\lambda_1 = 0$

$x_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$= \boxed{c_1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} e^t \cos t \\ -e^t \sin t \\ 0 \end{bmatrix} + c_3\begin{bmatrix} e^t \sin t \\ e^t \cos t \\ 0 \end{bmatrix}}$$