Let A be a square matrix,
all entries $\boxed{\text{real}}$ numbers

Suppose λ, x : eigenvalue-eigenvector pair of A

So: $Ax = \lambda x$

Then: $\overline{Ax} = \overline{\lambda x}$

$\overline{a+0i} = a - 0i = a$

$\overline{a+0i} = \bar{a}$

So: $\bar{A}\,\bar{x} = \bar{\lambda}\,\bar{x}$

$A\bar{x} = \bar{\lambda}\bar{x}$

scalar vector

So: $\bar{\lambda}, \bar{x}$ are eigenvalue-eigenvector pair of A

Solving $y' = Ay$

General Solutions:

$$\boxed{c_1 e^{\lambda_1 t} x_1 + c_2 e^{\lambda_2 t} x_2} + c_3 e^{\lambda_3 t} x_3 + \dots + c_n e^{\lambda_n t} x_n$$

What if λ_1 complex?

$$\bar{\lambda}_1 = \lambda_2$$

$$\bar{x}_1 = x_2$$

$$\overline{\lambda_1 t} = \bar{\lambda}_1 \bar{t} = \bar{\lambda}_1 t$$

$$c_1 e^{\lambda_1 t} x_1 + c_2 e^{\overline{\lambda_1 t}} \bar{x}_1$$

$$= c_1 \underbrace{e^{\lambda_1 t} x_1} + c_2 \underbrace{\overline{e^{\lambda_1 t} x_1}}$$

Observation: $f + ig = \boxed{e^{\lambda_1 t} x_1}$

$$c_1(f+ig) + c_2\overline{(f+ig)}$$

$$= c_1(f+ig) + c_2(f-ig)$$

$$= \underbrace{(c_1+c_2)}_{\text{some constant}} f + \underbrace{i(c_1-c_2)}_{\text{some constant}} g$$

$$= af + bg \qquad a, b: \text{ some constants}$$

$$= a\,\mathrm{Re}(e^{\lambda_1 t} x_1) + b\,\mathrm{Im}(e^{\lambda_1 t} x_1)$$