Eigenvalues & Eigenvectors

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$$

Eigenvalues: λ where $\det(A - \lambda I) = 0$

$$\det\begin{pmatrix} 1-\lambda & 4 & 5 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 3-\lambda \end{pmatrix} = (1-\lambda)(2-\lambda)(3-\lambda) = 0$$

(upper-$\triangle$ar)

$\lambda_1 = 1$ $\quad$ $\boxed{\lambda_2 = 2}$ $\quad$ $\lambda_3 = 3$

$\boxed{\lambda_2 = 2}$ Find associated eigenvector

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 2\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ (∞ many)

$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is e'vector

$\Rightarrow s\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ also e'vector

(also solution)

$$\begin{cases} x + 4y + 5z = 2x \\ \quad 2y + 6z = 2y \\ \quad\quad 3z = 2z \end{cases} \quad] \quad \boxed{z = 0}$$

$$\begin{cases} x + 4y = 2x \\ 2y = 2y \end{cases} \longrightarrow \quad 4y = x$$

Solutions: $\begin{bmatrix} 4y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$

$\lambda_2 = 2, \quad k_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$

$X = a k_1 + b k_2 + c k_3$

$$\begin{bmatrix} 47 \\ 16 \\ 2 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix}$$

$$\begin{cases} a + 4b + 29c = 47 \\ b + 12c = 16 \\ \boxed{2c = 2} \end{cases}$$

Solve for a, b, c

In general: Gaussian elimination

This case easier: substitution

$c = 1$

$$\begin{cases} a + 4b + 29 = 47 \\ \boxed{b + 12 = 16} \end{cases} \quad b = 4$$

$$\{ a + 4(6) + 29 = 47$$

$a + 16 + 29 = 47$

$a + 45 = 47 \quad a = 2$

$$\begin{bmatrix} 47 \\ 16 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix}$$

$X \qquad k_1 \qquad k_2 \qquad k_3$

$$
\begin{aligned}
Ax &= A(2k_1 + 4k_2 + k_3) \\
&= 2(Ak_1) + 4(Ak_2) + Ak_3 \\
&= 2\cdot\lambda_1 k_1 + 4\cdot\lambda_2 k_2 + \lambda_3 k_3
\end{aligned}
$$

$$
\begin{aligned}
A^2 x &= A(2\cdot\lambda_1 k_1 + 4\lambda_2 k_2 + \lambda_3 k_3) \\
&= 2\lambda_1 (Ak_1) + 4\lambda_2 (Ak_2) + \lambda_3 (Ak_3) \\
&= 2\lambda_1 \cdot \lambda_1 k_1 + 4\lambda_2 \cdot \lambda_2 k_2 + \lambda_3 \cdot \lambda_3 k_3 \\
&= 2\cdot\lambda_1^2 k_1 + 4\cdot\lambda_2^2 k_2 + \lambda_3^2 k_3
\end{aligned}
$$

$$
\begin{aligned}
A^{900} x &= 2\cdot\lambda_1^{900} k_1 + 4\lambda_2^{900} k_2 + \lambda_3^{900} k_3 \\
&= 2(1)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4(2^{900})\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + 3^{900}\begin{bmatrix} 29 \\ 12 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 + 2^{904} + 29\cdot 3^{900} \\ 2^{902} + 12\cdot 3^{900} \\ 2\cdot 3^{900} \end{bmatrix}
\end{aligned}
$$

In general:

$$X = ak_1 + bk_2 + ck_3$$

$$A^n X = a \cdot \lambda_1^n k_1 + b \cdot \lambda_2^n k_2 + c \cdot \lambda_3^n k_3$$

$$X_n = P^n X_0$$

get X_0 as a linear combination of eigenvectors, k_1 & k_2

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \end{bmatrix} + b\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{aligned} a + 2b &= 1 \\ -a + 3b &= 0 \\ \hline 5b &= 1 \end{aligned}$$

so $\boxed{b = 1/5}$

$a + 2b = 1$
$a + 2/5 = 1$
$\boxed{a = 3/5}$

$$X_0 = \frac{3}{5}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$X_n = P^n X_0 = P^n\left(\frac{3}{5}k_1 + \frac{1}{5}k_2\right)$$

$$= (\lambda_1)^n \cdot \frac{3}{5}k_1 + (\lambda_2)^n \frac{1}{5}k_2$$

$$= \underbrace{\frac{1}{6^n}}_{\to 0} \cdot \frac{3}{5} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\lim_{n\to\infty} X_n = 0\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

What if x_0 were different?

$$x_0 = \begin{bmatrix} a \\ 1-a \end{bmatrix}$$

Write x_0 as lin comb of k_1 & k_2 (e'vectors)

$$\begin{bmatrix} a \\ 1-a \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{aligned} x + 2y &= a \\ -x + 3y &= 1-a \\ \hline 5y &= 1 \end{aligned} \rightarrow \boxed{y = 1/5}$$

$$x + 2/5 = a \qquad \boxed{x = a - 2/5}$$

$$x_0 = (a - 2/5)\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{5}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$x_n = P^n x_0 = (a-2/5)\cdot \underbrace{\left(\frac{1}{6}\right)^n}_{\to 0}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{5}(1)^n\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\lim_{n\to\infty} P^n x_0 = 0 + \frac{1}{5}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

Smarty-Pants Method (shorter, clever)

X_0 = any reasonable thing

$X_0 = x\begin{bmatrix}1\\-1\end{bmatrix} + y\begin{bmatrix}2\\3\end{bmatrix}$ ← not find x, y - leave them for now

$$X_n = P^n X_0 = P^n\left(x\begin{bmatrix}1\\-1\end{bmatrix} + y\begin{bmatrix}2\\3\end{bmatrix}\right)$$

$$= x\cdot\lambda_1^n\begin{bmatrix}1\\-1\end{bmatrix} + y\cdot\lambda_2^n\begin{bmatrix}2\\3\end{bmatrix}$$

$$= x\underbrace{\left(\frac{1}{6^n}\right)}_{\to 0}\begin{bmatrix}1\\-1\end{bmatrix} + y\cdot 1\cdot\begin{bmatrix}2\\3\end{bmatrix}$$

$$\lim_{n\to\infty} X_n = 0 + y\begin{bmatrix}2\\3\end{bmatrix}$$

now I need to know what y is

$y\begin{bmatrix}2\\3\end{bmatrix} = \begin{pmatrix}2y\\3y\end{pmatrix}$ is a probability vector

So: $2y + 3y = 1$

So $y = 1/5$

$$\begin{bmatrix}2/5\\3/5\end{bmatrix}$$