Course Notes 6.2: Eigenanalysis Simplifies Matrix Powers

Outline

Week 11: Eigenvalues and eigenvectors: complex numbers and random walks

Course Notes: 6.2

Goals: More practice finding eigenvalues and eigenvectors; expanding these to the complex numbers; using them in the context of random walks.
Random Walks - Review

• \( \mathbf{x}_n = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_k \end{bmatrix} \) and \( p_1 + p_2 + \cdots + p_k = 1 \)

Probability vector, at time \( n \)
Random Walks - Review

- $x_n = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_k \end{bmatrix}$ and $p_1 + p_2 + \cdots + p_k = 1$

  Probability vector, at time $n$

- $x_n = Px_{n-1}$
Random Walks - Review

- \( x_n = \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_k \end{bmatrix} \) and \( p_1 + p_2 + \cdots + p_k = 1 \)

  Probability vector, at time \( n \)

- \( x_n = Px_{n-1} \)

- \( x_n = P^n x_0 \)
Random Walks and Eigenvalues

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Your initial state is $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. What happens after many tests?

$$P^n x_0 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} P^n \begin{bmatrix} 1 \\ 5 \end{bmatrix} P^n x_0 = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \left(\frac{1}{6}\right)^n \begin{bmatrix} 1 \\ 5 \end{bmatrix} P^n x_0 = \frac{1}{5} \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}$$

What if $x_0$ were different?
Random Walks and Eigenvalues

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$\lambda_1 = \frac{1}{6}$, $k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$;  
$\lambda_2 = 1$, $k_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
Random Walks and Eigenvalues

\[ \lambda_1 = \frac{1}{6}, \quad k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad \lambda_2 = 1, \quad k_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

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### Random Walks and Eigenvalues

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\) |      |

\[ \lambda_1 = \frac{1}{6}, \quad k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]; \quad \lambda_2 = 1, \quad k_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

Your initial state is \( x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). What happens after many tests?

\[ x_0 = \frac{3}{5} k_1 + \frac{1}{5} k_2 \]
Random Walks and Eigenvalues

$$\begin{array}{c|cc}
\text{from} & \text{pass} & \text{fail} \\
\hline
\text{to} & \frac{1}{2} & \frac{1}{3} \\
\text{pass} & \frac{1}{2} & \frac{2}{3} \\
\text{fail} & & \\
\end{array}$$

$$\lambda_1 = \frac{1}{6}, \ k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad \lambda_2 = 1, \ k_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

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$$x_0 = \frac{3}{5}k_1 + \frac{1}{5}k_2$$

$$P^n x_0 = \frac{3}{5} P^n k_1 + \frac{1}{5} P^n k_2 = \frac{3}{5} \left( \frac{1}{6} \right)^n k_1 + \frac{1}{5} k_2$$
Random Walks and Eigenvalues

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\[ \lambda_1 = \frac{1}{6}, \ k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad \lambda_2 = 1, \ k_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

Your initial state is \( x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). What happens after many tests?

\[ x_0 = \frac{3}{5} k_1 + \frac{1}{5} k_2 \]

\[ P^n x_0 = \frac{3}{5} P^n k_1 + \frac{1}{5} P^n k_2 = \frac{3}{5} \left( \frac{1}{6} \right)^n k_1 + \frac{1}{5} k_2 \]

\[ \lim_{n \to \infty} P^n x_0 = \frac{1}{5} k_2 = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix} \]
Random Walks and Eigenvalues

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\( \lambda_1 = \frac{1}{6}, \ k_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}; \quad \lambda_2 = 1, \ k_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \)

Your initial state is \( x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). What happens after many tests?

\[
\begin{align*}
x_0 &= \frac{3}{5} k_1 + \frac{1}{5} k_2 \\
P^n x_0 &= \frac{3}{5} P^n k_1 + \frac{1}{5} P^n k_2 = \frac{3}{5} \left( \frac{1}{6} \right)^n k_1 + \frac{1}{5} k_2 \\
\lim_{n \to \infty} P^n x_0 &= \frac{1}{5} k_2 = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix}
\end{align*}
\]

What if \( x_0 \) were different?
Theorem

If $P$ is a transition matrix (non-negative entries with all columns summing to one) that in addition has all positive entries then $P$ has an eigenvalue 1 with a single eigenvector $k_1$ that can be chosen to be a probability vector. All other eigenvalues satisfy $|\lambda| < 1$ with eigenvectors with components that sum to zero. Thus,

$$\lim_{n \to \infty} x_n = k_1$$

for any $x_0$. That is, $k_1$ is an equilibrium probability.
Eigenvalues of Probability Transition Matrices

**Theorem**

If $P$ is a transition matrix (non-negative entries with all columns summing to one) that in addition has all positive entries then $P$ has an eigenvalue 1 with a single eigenvector $k_1$ that can be chosen to be a probability vector. All other eigenvalues satisfy $|\lambda| < 1$ with eigenvectors with components that sum to zero. Thus,

$$\lim_{n \to \infty} x_n = k_1$$

for any $x_0$. That is, $k_1$ is an equilibrium probability.

[Proof, of sorts]

In short: that last example was typical. As long as a probability matrix has no zeroes:

- Probability matrices have 1 as an eigenvalue
- There will be some equilibrium that the system will reach in the long run, regardless of initial state, corresponding to an eigenvector of 1.
Equilibrium Probability

Which of these random walk models seems likely to have an equilibrium probability? Is it clear what would it be?

In any given day, your odds of dying are 1 in 1,000.

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<tr>
<th></th>
<th>alive</th>
<th>dead</th>
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<tr>
<td>alive</td>
<td>0.999</td>
<td>0</td>
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<tr>
<td>dead</td>
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Equilibrium probability: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$; everybody dies.
Equilibrium Probability

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Choosing a career:

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No equilibrium probability—depends on initial stages.
Equilibrium Probability

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Region of residence.

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Equilibrium Probability

Which of these random walk models seems likely to have an equilibrium probability? Is it clear what would it be?

In any given day, your odds of dying are 1 in 1,000.

\[
\begin{array}{cc}
\text{alive} & \text{dead} \\
\hline
\text{alive} & 0.999 & 0 \\
\text{dead} & 0.001 & 1 \\
\end{array}
\]

Choosing a career:

\[
\begin{array}{ccc}
? & \text{math} & \text{eng} \\
? & 0.3 & 0 & 0 \\
\text{math} & 0.2 & 1 & 0 \\
\text{eng} & 0.5 & 0 & 1 \\
\end{array}
\]

Region of residence.

\[
\begin{array}{cc}
\text{N/S Am.} & \text{Other} \\
\hline
\text{N/S Am.} & .99 & .01 \\
\text{Other} & .01 & .99 \\
\end{array}
\]

Equilibrium probability: \[
\begin{bmatrix}
0.5 \\
0.5
\end{bmatrix}; \text{long-term average.}
\]
Equilibrium Probability

Find the equilibrium probability of the system.

In a relationship or not, by year.

<table>
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regardless of $x_0$
Equilibrium Probability

Find the equilibrium probability of the system.

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**Theorem**

If $P$ is a transition matrix that in addition has all positive entries then $P$ has an eigenvalue 1 with a single eigenvector $k_1$ that can chosen to be a probability vector; in this case $k_1$ is the equilibrium probability.
Equilibrium Probability

Find the equilibrium probability of the system.

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Eigenvalues and eigenvectors:

\[
\begin{align*}
\lambda_1 &= 1 \\
\lambda_2 &= \frac{3}{20} \\
k_1 &= \begin{bmatrix} 1 \\ \frac{12}{5} \end{bmatrix} \\
k_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{align*}
\]
Equilibrium Probability

Find the equilibrium probability of the system.

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Eigenvalues and eigenvectors:

\[
\lambda_1 = 1, \quad k_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \\
\lambda_2 = \frac{3}{20}, \quad k_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

For any \( x_0 = a_1 k_1 + a_2 k_2 \):

\[
P^n x_0 = a_2 k_1 + a_2 \left( \frac{3}{20} \right)^n k_2 \quad \overset{n \to \infty}{\longrightarrow} \quad a_2 k_1
\]
Equilibrium Probability

Find the equilibrium probability of the system.

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Eigenvalues and eigenvectors:

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\begin{align*}
\lambda_1 &= 1 \\
\lambda_2 &= \frac{3}{20}
\end{align*}
\]

\[
\begin{bmatrix}
1 \\
12
\end{bmatrix}, \begin{bmatrix}
1 \\
-1
\end{bmatrix}
\]

For any \(x_0 = a_1 k_1 + a_2 k_2:\)

\[
P^n x_0 = a_2 k_1 + a_2 \left(\frac{3}{20}\right)^n k_2 \xrightarrow{n \to \infty} a_2 k_1
\]

\[
\lim_{n \to \infty} P^n x_0 = \begin{bmatrix}
5/17 \\
12/17
\end{bmatrix}
\]

regardless of \(x_0\)
Computation Practice

\[ P = \begin{bmatrix} 1/3 & 1/2 \\ 2/3 & 1/2 \end{bmatrix} \quad x_0 = \begin{bmatrix} a \\ 1 - a \end{bmatrix}, \ a \in [0, 1] \]

1. Find all eigenvalues of \( P \), and an associated eigenvector to each.

2. Write \( x_0 \) as a linear combination of eigenvectors of \( P \).

3. Calculate \( x_n \), where \( n \) is some positive integer.

4. Find the equilibrium probability of \( P \).
1. Find Eigenvalues and Eigenvectors

By our theorem, we know that 1 will be an eigenvalue. However, for the sake of practice, let’s find them the old-fashioned way.

Eigenvalues of $P$ are precisely those scalars $\lambda$ such that $\det(P - \lambda I) = 0$. So we set the determinant equal to zero:

$$\det \begin{bmatrix} 1/3 - \lambda & 1/2 \\ 2/3 & 1/2 - \lambda \end{bmatrix} = \left( \frac{1}{3} - \lambda \right) \left( \frac{1}{2} - \lambda \right) - \left( \frac{1}{2} \right) \left( \frac{2}{3} \right) = \lambda^2 - \frac{5}{6} \lambda - \frac{1}{6}$$

And find $\lambda_1 = 1$ (as expected) and $\lambda_2 = -\frac{1}{6}$.

To find the associated eigenvectors, we set $Px = \lambda x$. (Next Slide)
1. Find Eigenvalues and Eigenvectors

\[ \lambda_1 = 1 \]

\[
\begin{bmatrix}
\frac{1}{3} & \frac{1}{2} \\
\frac{2}{3} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 1
\begin{bmatrix}
x \\
y
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\frac{-2}{3} & \frac{1}{2} \\
\frac{2}{3} & -\frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

The solutions to this system are of the form \( s \begin{bmatrix} 3 \\ 4 \end{bmatrix} \) for some scalar \( s \). Any vector of this form will do.

\[ \lambda_2 = -\frac{1}{6} \]

\[
\begin{bmatrix}
\frac{1}{3} & \frac{1}{2} \\
\frac{2}{3} & \frac{1}{2}
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= -\frac{1}{6}
\begin{bmatrix}
x \\
y
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{2}{3} & \frac{2}{3}
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

The solutions to this system are of the form \( s \begin{bmatrix} 1 \\ -1 \end{bmatrix} \) for some scalar \( s \). Any vector of this form will do.
2. Basis Vectors

To find $x_0$ as a combination of eigenvectors, we have to CHOOSE our eigenvectors. I like integers, so I’ll use $k_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $k_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Your vectors may be scalar multiples of these. The equation we have to solve is:

$$\begin{bmatrix} a \\ 1 - a \end{bmatrix} = x \begin{bmatrix} 3 \\ 4 \end{bmatrix} + y \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

which can be rewritten as

$$\begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ 1 - a \end{bmatrix}$$

Since $a$ is a constant, we can solve this using an augmented matrix and row reduction.

$$\begin{bmatrix} 3 & 1 & a \\ 4 & -1 & 1 - a \end{bmatrix} \xrightarrow{R_2 \rightarrow R_2 + R_1} \begin{bmatrix} 3 & 1 & a \\ 7 & 0 & 1 \end{bmatrix}$$

So $x = \frac{1}{7}$ and $y = a - \frac{3}{7}$. That is, $x_0 = \frac{1}{7}k_1 + (a - \frac{3}{7})k_2$. 
3. Find $x_n$

Recall $x_n = P^n x_0$. With our previous work, the answer is an easy calculation.

\[
x_n = P^n x_0 = P^n \left( \frac{1}{7} k_1 + \left( \frac{3}{7} - a \right) k_2 \right)
\]
\[
= \frac{1}{7} P^n k_1 + \left( \frac{3}{7} - a \right) P^n k_2
\]
\[
= \frac{1}{7} k_1 + \left( \frac{3}{7} - a \right) \left( \frac{1}{6} \right)^n k_2
\]
\[
= \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix} + \begin{bmatrix} \left( \frac{3}{7} - a \right) (-1/6)^n \\ - \left( \frac{3}{7} - a \right) (-1/6)^n \end{bmatrix}
\]
\[
= \begin{bmatrix} \frac{3}{7} + \left( \frac{3}{7} - a \right) (-1/6)^n \\ \frac{4}{7} - \left( \frac{3}{7} - a \right) (-1/6)^n \end{bmatrix}
\]
4. Find the Equilibrium Probability

Recall the equilibrium probability is \( \lim_{n \to \infty} x_n = \lim_{n \to \infty} P^n x_0 \). With our previous work, the answer is an easy calculation. Using an intermediate result from the last slide:

\[
\lim_{n \to \infty} x_n = \lim_{n \to \infty} \left[ \frac{1}{7} k_1 + \left( \frac{3}{7} - a \right) \left( \frac{1}{6} \right)^n k_2 \right]
\]

\[
= \frac{1}{7} k_1
\]

\[
= \begin{bmatrix} \frac{3}{7} \\ \frac{4}{7} \end{bmatrix}
\]

ALTERNATELY, our theorem tells us that the equilibrium probability will always be an eigenvector associated with the eigenvalue \( \lambda = 1 \). Since our eigenvectors were of the form \( s[3, 4] \), we can find the equilibrium probability by figuring out which value of \( s \) gives us a vector whose entries sum to one; \( s = 7 \) is that scalar.
Complex Eigenvalues:

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

Calculate \( A^{90}x \).
Complex Eigenvalues:

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

Calculate \( A^{90}x \).

\[ \lambda_1 = i, \, k_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \]

\[ \lambda_2 = -i, \, k_2 = \begin{bmatrix} i \\ 1 \end{bmatrix} \]
Complex Eigenvalues:

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

Calculate \( A^{90} \mathbf{x} \).

\[ \lambda_1 = i, \mathbf{k}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \lambda_2 = -i, \mathbf{k}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix} \]

\[ \mathbf{x} = (1.5 + i)\mathbf{k}_1 + (1.5 - i)\mathbf{k}_2 \]
Complex Eigenvalues:

\[
A = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \quad x = \begin{bmatrix}
2 \\
3
\end{bmatrix}
\]

Calculate \(A^{90}x\).

\[
\lambda_1 = i, \quad k_1 = \begin{bmatrix}
-i \\
1
\end{bmatrix} \quad \lambda_2 = -i, \quad k_2 = \begin{bmatrix}
i \\
1
\end{bmatrix}
\]

\[
x = (1.5 + i)k_1 + (1.5 - i)k_2
\]

\[
A^{90}x = (1.5 + i)A^{90}k_1 + (1.5 - i)A^{90}k_2
\]

\[
= (1.5 + i)(i)^{90}k_1 + (1.5 - i)(-i)^{90}k_2
\]

\[
= (1.5 + i)(-1)k_1 + (1.5 - i)(-1)k_2 = \begin{bmatrix}
-2 \\
-3
\end{bmatrix}
\]
Complex Eigenvalues: A Neat Trick

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

\[ \lambda_1 = i, \mathbf{k}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \lambda_2 = -i, \mathbf{k}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix} \]
Complex Eigenvalues: A Neat Trick

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

\[ \lambda_1 = i, k_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \quad \lambda_2 = -i, k_2 = \begin{bmatrix} i \\ 1 \end{bmatrix} \]

If the entries of \( A \) are all real, its eigenvalues and eigenvectors are complex conjugates of one another.
**Complex Eigenvalues: A Neat Trick**

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad x = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \]

\[ \lambda_1 = i, \quad k_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix} \]

\[ \lambda_2 = -i, \quad k_2 = \begin{bmatrix} i \\ 1 \end{bmatrix} \]

If the entries of \( A \) are all real, its eigenvalues and eigenvectors are complex conjugates of one another.

\[ A x = \lambda x \quad \Leftrightarrow \quad \overline{A x} = \overline{\lambda x} \quad \Leftrightarrow \quad \overline{A x} = \overline{\lambda x} \quad \Leftrightarrow \quad A \overline{x} = \overline{\lambda x} \]
Complex Eigenvalues: A Neat Trick

Find all eigenvalues and eigenvectors of:

\[ A = \begin{bmatrix} -3 & 5 \\ -2 & -1 \end{bmatrix} \]
Complex Eigenvalues: A Neat Trick

Find all eigenvalues and eigenvectors of:

\[ A = \begin{bmatrix} -3 & 5 \\ -2 & -1 \end{bmatrix} \]

\[
\text{det} \left( \begin{bmatrix} -3 & 5 \\ -2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \text{det} \begin{bmatrix} -3 - \lambda & 5 \\ -2 & -1 - \lambda \end{bmatrix} = \lambda^2 + 4\lambda + 13
\]

Roots:

\[
-4 \pm \frac{\sqrt{16 - 4(13)}}{2} = -2 \pm 3i
\]
Complex Eigenvalues: A Neat Trick

Find all eigenvalues and eigenvectors of:

\[ A = \begin{bmatrix} -3 & 5 \\ -2 & -1 \end{bmatrix} \]

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\det \left( \begin{bmatrix} -3 & 5 \\ -2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \det \begin{bmatrix} -3 - \lambda & 5 \\ -2 & -1 - \lambda \end{bmatrix} = \lambda^2 + 4\lambda + 13
\]

Roots:

\[
-4 \pm \frac{\sqrt{16 - 4(13)}}{2} = -2 \pm 3i
\]

\[ \lambda_1 = -2 - 3i, \]
Complex Eigenvalues: A Neat Trick

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\]

Roots:

\[
\frac{-4 \pm \sqrt{16 - 4(13)}}{2} = -2 \pm 3i
\]

\[ \lambda_1 = -2 - 3i, \quad x_1 = \begin{bmatrix} 1 + 3i \\ 2 \end{bmatrix} \]
Complex Eigenvalues: A Neat Trick

Find all eigenvalues and eigenvectors of:

\[ A = \begin{bmatrix} -3 & 5 \\ -2 & -1 \end{bmatrix} \]

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\text{det} \left( \begin{bmatrix} -3 & 5 \\ -2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) = \text{det} \begin{bmatrix} -3 - \lambda & 5 \\ -2 & -1 - \lambda \end{bmatrix} = \lambda^2 + 4\lambda + 13
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Roots:
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Complex Eigenvalues: A Neat Trick

Find all eigenvalues and eigenvectors of:

\[ A = \begin{bmatrix} -3 & 5 \\ -2 & -1 \end{bmatrix} \]

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\]

Roots: \( \frac{-4 \pm \sqrt{16 - 4(13)}}{2} = -2 \pm 3i \)

\( \lambda_1 = -2 - 3i, \quad x_1 = \begin{bmatrix} 1 + 3i \\ 2 \end{bmatrix} \)

\( \lambda_2 = -2 + 3i, \quad x_2 = \begin{bmatrix} 1 - 3i \\ 2 \end{bmatrix} \)
Let $A$ be a matrix with eigenvalue $\lambda$ and associated eigenvector $\mathbf{k}$.

True or False:

1. $2\mathbf{k}$ is an eigenvector of $A$, associated with $\lambda$.
2. It is possible that $\mathbf{k}$ is an eigenvalue of $A$ associated with a different eigenvalue (that is, other than $\lambda$).
3. All eigenvectors of $A$ associated with $\lambda$ are scalar multiples of $\mathbf{k}$.
4. $\mathbf{k}$ might be the zero vector.
5. $\lambda$ might be zero.
6. If $A$ has only real entries, then $\lambda$ is real.
7. If $A$ has only real entries, and $\mathbf{k}$ has only real entries, then $\lambda$ is real.
Eigenfact or Eigenfiction?

Let $A$ be a matrix with eigenvalue $\lambda$ and associated eigenvector $k$. True or False:

1. $2k$ is an eigenvector of $A$, associated with $\lambda$. **True**
2. It is possible that $k$ is an eigenvalue of $A$ associated with a different eigenvalue (that is, other than $\lambda$). **False**
3. All eigenvectors of $A$ associated with $\lambda$ are scalar multiples of $k$. **False**
4. $k$ might be the zero vector. **False**
5. $\lambda$ might be zero. **True**
6. If $A$ has only real entries, then $\lambda$ is real. **False**
7. If $A$ has only real entries, and $k$ has only real entries, then $\lambda$ is real. **True**
Let $A$ be a matrix with eigenvalue $\lambda$ and associated eigenvector $k$.

True or False:

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True or False:

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   Answer: False

4. $k$ might be the zero vector.  
   Answer: False

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