Math 105, Section 208, Quiz 2
Tuesday, January 23

Communicating mathematics is an important skill to practice, so simplify and justify all answers unless otherwise directed. Show your work, and use proper notation.

Second Derivative Test:
Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y)$, where $f(x, y)$ is a function whose second partial derivatives are continuous throughout an open disk centred at the point (a, b), and $f_x(a, b) = f_y(a, b) = 0$.

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b).
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b).
3. If $D(a, b) < 0$, then f has a saddle point at (a, b).

1. Find both values of $\mathbf{a}$ such that the function $\mathbf{f(x,y) = x^2 + y^2}$ over the region $\mathbf{x^2 + (y-a)^2 \leq 1}$ has an absolute maximum value of 9, and an absolute minimum value of 1. $\boxed{5}$

Hint: recall that the equation $x^2 + (y-a)^2 = 1$ describes a circle of radius 1, centred at $(0, a)$.

Solution: Absolute extrema over a closed, bounded region occur at CPs and along the boundary.

To find CPs, we differentiate.
$f_x = 2x$, and $f_y = 2y$, so the only CP is at $(0,0)$. At this point, $f(0,0) = 0$.

Now, let's consider the boundary. Along the boundary, $x^2 = 1-(y-a)^2$, so our function is:

$$f(x,y) = 1-(y-a)^2 + y^2 = 1 - y^2 + 2ay - a^2 - y^2 = 1 - a^2 + 2ay$$

This is a line, depending on y, so its extreme values will occur at the largest and smallest values of y. (Note that a might be positive or negative.) Since our boundary is a circle of radius 1 centred at $(0,a)$, along the boundary, $a-1 \leq y \leq a+1$. This tells us our extreme values along the boundary in terms of a:

If $y = a-1$, then $f = 1-a^2+2ay = 1-a^2+2a(a-1) = 1-a^2+2a^2-2a = a^2-2a+1 = (a-1)^2$.
If $y = a+1$, then $f = 1-a^2+2ay = 1-a^2+2a(a+1) = 1-a^2+2a^2+2a = a^2+2a+1 = (a+1)^2$.

So, we're looking for values of a with $(a-1)^2 = 9$ and $(a+1)^2 = 1$, or with $(a-1)^2 = 1$ and $(a+1)^2 = 9$. These solutions are $a = 2$ and $a = -2$.

Note for these values of a, $(0,0)$ is not in our region, so we don't have any CPs in the region.

Marking scheme:

Everything correct: 5 points

Correct answer with some work, but not a thorough justification (for example, no explanation of why they checked only points with $x = 0$ or $y = 0$): 3 points

Finding (0,0) as only CP and successfully plugging in the boundary to $f(x,y)$ to make a function of only y: 2 points

Finding (0,0) as CP and no other: 1 point

Alternate Method

Some students used Lagrange here. In this case, $g(x,y) = x^2 + (y-a)^2 - 1 = 0$, so we need to solve this system of equations to find potential locations of extrema over the boundary:

1. $2x = \lambda 2x$
2. $2y = 2\lambda(y-a)$
3. $x^2 + (y-a)^2 - 1 = 0$

From (1), we conclude $x = 0$ or $\lambda = 1$.
In the case $x = 0$, from (3) we conclude $y = a \pm 1$. This tells us we should check the points $(0, a+1)$ and $(0, a-1)$.
In the case $\lambda = 1$, from (2) we conclude $a = 0$, which tells us $(0,0)$ is in R. Then the absolute minimum of our function in R is 0, which is not what we're going for. So, there are no points to check from this case.

Now, we compare:

- $f(0, a+1) = (a+1)^2$
- $f(0, a-1) = (a-1)^2$

So, our extrema will be $(a+1)^2$ and $(a-1)^2$. We finish as in the other method.

Marking scheme: As before, but the marks for plugging in are now marks for finding the correct four points to check.

2. Find and classify all critical points of the function $\mathbf{f(x,y) = x^4 + (x+2y+3)^4}$. Remember to justify your answer completely. $\boxed{5}$

Solution: $f_x = 4x^3 + 4(x+2y+3)^3$
$f_y = 8(x+2y+3)^3$;
If $f_y = 0$, then $x + 2y + 3 = 0$; if additionally $f_x = 0$, then $x = 0$, and so $y = -3/2$. So, the only critical point is $(0, -3/2)$.

If we try to use the second derivative test, we'll get an inconclusive result. However, we don't need it to classify this critical point. Since both summands of our function are raised to even powers, the smallest either of them could be is 0, so the smallest the function could ever be is 0. Since $f(0, -3/2) = 0$, we see our critical point is the location of a local minimum.

Marking scheme:
First partial derivatives: 1 point
Finding the CP: 1 point
Finding no other CPs: 1 points
Classifying CP: 2 points If a student made a mistake, for instance a wrong partial derivative, points will depend on whether or not it made the problem easier.
Correct work must be shown for credit. For example, if a student simply puts CPs where $x = 0$ and $y = 0$, they don't get credit for finding the CP. If they classify the CP, but don't indicate their reasoning, no credit.

3. Double or Nothing **(optional)**

Pick one of the two questions on this quiz. Full marks will be doubled; partial marks will be deleted.

The purpose of this question is to encourage you to develop self-reliance in evaluating your understanding.