Math 105, Section 204, Quiz 2
Wednesday, January 24

Communicating mathematics is an important skill to practice, so simplify and justify all answers unless otherwise directed. Show your work, and use proper notation.

Second Derivative Test:
Let $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - f_{xy}^2(x, y)$, where $f(x, y)$ is a function whose second partial derivatives are continuous throughout an open disk centred at the point (a, b), and $f_x(a, b) = f_y(a, b) = 0$.
1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then f has a local maximum value at (a, b).
2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then f has a local minimum value at (a, b).
3. If $D(a, b) < 0$, then f has a saddle point at (a, b).

1. A surface is described by the equation $\mathbf{z = (x+1)^2 y^3 + y^2 + 2y + 4}$ where z is the vertical axis (as usual). You drop a ball onto the surface, and it settles into the point (a, b). Give all possible values of (a, b). $\boxed{5}$

> **Solution:** A ball could only settle at a critical point, where the surface isn't tilted. So, we find all critical points.
>
> $z_x = 2(x+1)y^3$; for this to be zero, we need $x = -1$ or $y = 0$.
> $z_y = 3(x+1)^2y^2 + 2y + 2$; if $y = 0$ (as suggested above) then $z_y \neq 0$, so no CP has $y = 0$. Therefore, any CP has $x = -1$. From z_y, we see this implies $y = -1$. So, our only CP is $(-1, -1)$, so this is the only place the ball could settle.

2. Someone is trying to sketch the level curves of the surface defined by the equation $\mathbf{z = e^{x^2} - 2y^2}$ over the closed region $\mathbf{R} = \{(\mathbf{x}, \mathbf{y}) : \mathbf{x^2 + y^2 \leq 4}\}$. They notice that some constant values of z give no solutions over R. $\boxed{5}$

   Find the range constant values of z that will have solutions in R. That is, find the largest and smallest values of z so that there exists a point (x, y) in R with $z = e^{x^2} - y^2$. [sic]

   Hint: by the chain rule, $\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{x^2}\right] = 2xe^{x^2}$.

> **Solution:** The question is asking for largest and smallest values of the function $f(x, y) = e^{x^2} - 2y^2$ over the closed region bounded by a circle centred at the origin with radius 2. So, it's asking for the absolute max and min over a bounded region. These occur at CPs and along boundaries.
>
> To find CPs, we differentiate.
> $z_x = 2xe^{x^2}$, so any CP has $x = 0$
> $z_y = -4y$, so any CP has $y = 0$
> So, the only CP is $(0, 0)$, and at this point $z = e^0 - 0 = 1$.
>
> Now, we check the boundary. Along the boundary, $y^2 = 4 - x^2$, so our function gives values $z = e^{x^2} - 2(4 - x^2)$; that is, $z = e^{x^2} + 2x^2 - 4$. This function grows larger for larger values of x^2, and smaller for smaller value of x^2. So, the extreme values of z along the boundary will occur for $x^2 = 0$ and $x^2 = 4$. These values are $e^4 + 2 \cdot 4 - 4 = e^4 + 4$, and $e^0 + 0 - 4 = -3$. (Note that 1 is between these two values, so the CP doesn't lead to an extremum.)
>
> So, the range of z values over R is from $z = -3$ to $z = e^4 + 4$.

3. Double or Nothing **(optional)**

   Pick one of the two questions on this quiz. Full marks will be doubled; partial marks will be deleted.

   The purpose of this question is to encourage you to develop self-reliance in evaluating your understanding.