Ch 9.2 Properties of Power Series

Def A <u>power series</u> has the form:

$$\sum_{k=0}^{\infty} c_k (x-a)^k$$

where $\{c_k\}$ constants, "coefficients"
a constant "centre"
x variable

– Infinite analogue of a polynomial.

$$c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots$$

Observed: Any polynomial defined for <u>all</u> real #s.

$\sum_{k=0}^{\infty} x^k$ power series: $c_k = 1$, $a = 0$ Only converges if $-1 < x < 1$

$\sum (1)(x-0)^k$

Domain: $(-1, 1)$

eg $\displaystyle\sum_{k=0}^{\infty} 2^{k+1}(x-3)^k$ $\boxed{\text{power series}}$

$c_k = 2^{k+1}$

$a = 3$

$\downarrow$

$2 + 2^2(x-3) + 2^3(x-3)^2 + \cdots$

looks like polynomial

eg $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{x^{2k}+1}$ $\boxed{\text{NOT a power series}}$

$1 + \frac{x}{x^2+1} + \frac{x^2}{x^4+1} + \cdots$

not look like polynomial

$\displaystyle\sum_{k=2}^{\infty} 2^{k+1}(x-3)^k = 2^3(x-3)^2 + \cdots$

$= 0 + 0(x-3) + 2^3(x-3)^2 + \cdots$

eg $\sum_{k=0}^{\infty} (2x+1)^k = \sum_{k=0}^{\infty} \left[2(x+\tfrac{1}{2})\right]^k = \sum_{k=0}^{\infty} 2^k(x-(-\tfrac{1}{2}))^k$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad c_k \qquad a$

IS a power series $\qquad \sum c_k (x-a)^k$

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k$$

"domain" is the idea behind "interval of convergence"

The set of values x for which a power series converges is its <u>interval of convergence</u>. Its <u>radius of convergence</u>, R, is distance from centre of series to the boundary of its interval of convergence.

eg $$f(x) = \sum_{k=0}^{\infty} x^k$$

geometric series

Power series

coeffs: $c_k = 1$

Centre: $a = 0$

$$\sum c_k (x-a)^k = \sum 1 (x-a)^k = \sum x^k$$

$$f(1/2) = \frac{1}{1-1/2}$$

$$f(-3/4) = \frac{1}{1-(-3/4)} = \frac{1}{1+3/4}$$

$$\to \sum_{k=0}^{\infty} \left(\frac{-3}{4}\right)^k$$

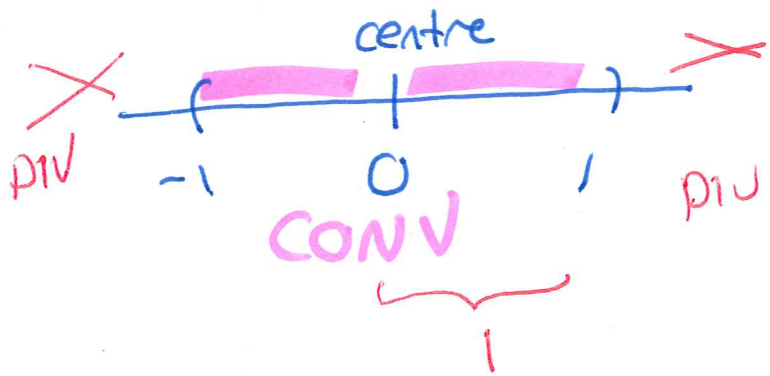
$$\sum r^n \to \frac{1}{1-r}$$

$$f(2) \overset{?}{=} \cancel{\frac{1}{1-2}}$$

$$= \sum_{k=0}^{\infty} (2^k) = 1+2+4+8+16+32+\dots$$ DIVERGES

Interval of Convergence: $(-1, 1)$

Radius of Conv $R = 1$



Idea: Radius of Conv

Fact: all power series converge at their centre |||

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1 (x-a)^1 + c_2 (x-a)^2 + c_3 (x-a)^3 + \dots$$

If $x=a$: $c_0 + c_1 (0)^1 + c_2 (0)^2 + c_3 (0)^3 + \dots$

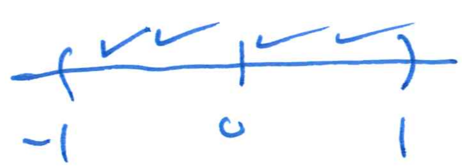
$= c_0$

$(0! = 1)$

$(0^0 = 1)$

Radius of Conv tells us how far away from the centre we can stray w/out leaving domain

$f(x) = \sum_{k=0}^{\infty} x^k$ makes sense as long as x is less than 1 away from its centre $(a=0)$



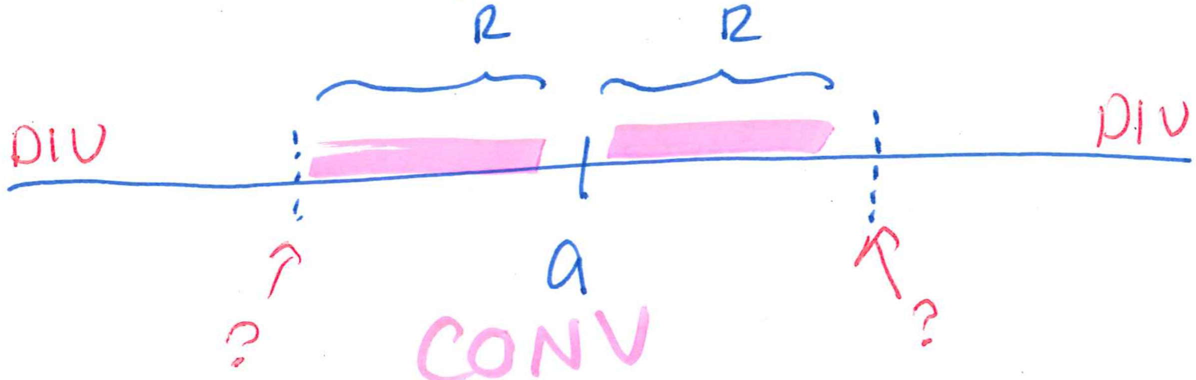
Theorem 9.3 p678

A power series centred at a converges in one of three ways:

1. The series converges for all real x.
   In that case: $R = \infty$

2. There is a real number $R > 0$ such that the series converges for all $|x-a| < R$ AND diverges for all $|x-a| > R$

dist btw x & a



3. The series converges only at $x = a$ (nowhere else)

In that case: $R = 0$

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ex Find radius of convergence of

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \left(\frac{1}{k!}\right)(x-0)^k$$

coeffs   centre

Ratio test (leave x undetermined)

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} \Big/ \frac{x^n}{n!} = \lim_{n\to\infty} \frac{\cancel{x^{n+1}}}{\cancel{x^n}} \cdot \frac{\cancel{n!}}{(n+1)\cancel{!}}$$

$$= \lim_{n\to\infty} \frac{x}{n+1} = 0 < 1$$

for every x

Radius of conv: $R = \infty$

ex $\sum_{k=0}^{\infty} 2^k (x-1)^k$ Radius of Conv?

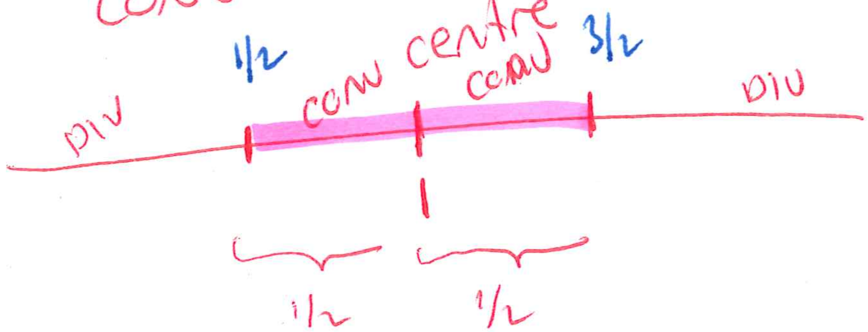
Ratio test

$r = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-1)^{n+1}}{2^n(x-1)^n}\right| = \cancel{\lim_{n\to\infty} 2(x-1)}$

$\lim_{n\to\infty}|2(x-1)| = \underbrace{2|x-1|}_{r}$ → <1: series conv; → >1: series diverges

$2|x-1|<1$
ie $|x-1|<1/2$:
CONV

$2|x-1|>1$
$|x-1|>1/2$:
DIV



Radius of conv: $1/2 = R$

Interpret: $\sum_{k=0}^{\infty} 2^k\left(\frac{1}{4}-1\right)^k$ CONVERGES

$(x = 1/4)$

(x=

$$\sum 2^k\left((1+1/4)-1\right)^k$$ converges

$(x = 1+1/4)$

$$= \sum 2^k\left(\tfrac{1}{4}\right)^k = \sum \left(\tfrac{1}{2}\right)^k$$

$(x = 2)$ DIVERGE

$$\sum 2^k(2-1)^k = \sum 2^k$$

$x = 0$

$$\sum 2^k(0-1)^k = \sum (-2)^k$$ DIV

ex Find radius of conv of

$$\sum_{k=0}^{\infty} \frac{(x-2)^{2k+1}}{2k+1}$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(x-2)^{2(n+1)+1}}{2(n+1)+1} \Big/ \frac{(x-2)^{2n+1}}{2n+1}$$

$$= \lim_{n\to\infty} \frac{(x-2)^{2n+3}}{(x-2)^{2n+1}} \cdot \frac{2n+1}{2n+3}$$

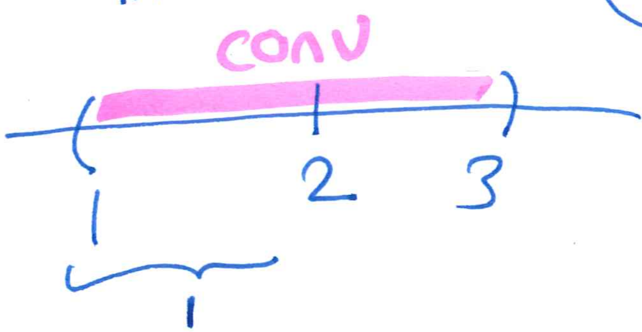
$$= \lim_{n\to\infty} (x-2)^2 \left(\frac{2n+1}{2n+3}\right)$$

$$= (x-2)^2 \longrightarrow$$ <1: series conv
>1: series div

$|x-2|^2 < 1$ conv

$|x-2| < 1$

$$\boxed{R=1}$$



Thm 9.5 p 680

Suppose the power series $\sum c_k (x-a)^k$ converges for $|x-a| < R$ and defines a function f on that interval.

1. f is differentiable ($\Rightarrow$ continuous) for $|x-a| < R$ and f' is found by differentiating series term-by-term, ie

$$f(x) = \sum c_k (x-a)^k$$

$$f'(x) = \sum c_k \, k(x-a)^{k-1}$$

2. The indefinite integral of f is found by integrating the power series term by term

$$\int f(x)\,dx = \int \sum c_k (x-a)^k \,dx$$

$$= \sum c_k \frac{(x-a)^{k+1}}{k+1}$$

ex $\dfrac{1}{1-x} = \displaystyle\sum_{k=0}^{\infty} x^k$, $\underline{|x| < 1}$

deriv deriv

So: $\dfrac{d}{dx}\left[(1-x)^{-1} \right] = -(1-x)^{-2}(-1) = \dfrac{1}{(1-x)^2}$

$$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} k x^{k-1}$$

ex $\displaystyle\int \frac{1}{1-x}\,dx = \int (-1)\frac{1}{u}\,du = -\ln|u| = -\ln|1-x| + C$

$u = 1-x$
$-du = dx$

$$= \int \left(\sum_{k=0}^{\infty} x^k \right) dx = \sum_{k=0}^{\infty} \frac{x^{k+1}}{k+1}$$

ex (a) Find a PS rep for

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

$$= \sum_{k=0}^{\infty} (-(x^2))^k$$

$$= \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$$\frac{1}{1-r} = \sum r^n$$

$$r = -(x^2)$$

$$|r|<1$$

$$|-x^2|<1$$

$$|x|<1$$

(b) Find a PS rep for arctan x

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k}$$

$$\Rightarrow \int \frac{1}{1+x^2}\,dx = \int \sum_{k=0}^{\infty} (-1)^k x^{2k}\,dx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}$$

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} + C$$

To find C, evaluate both sides at $x=0$:

$$\arctan(0) = \sum_{k=0}^{\infty} (-1)^k \frac{0^{2k+1}}{2k+1} + C$$

$$0 = 0 + 0 + 0 + \dots + C$$

So $C = 0$

$$\boxed{\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}}$$

$|x| < 1$

ex $\quad f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

(a) find $f'(x)$

(b) use that to guess what function $f(x)$ is

(dot dot dot notation helpful)

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$f'(x) = \sum_{k=0}^{\infty} \frac{k x^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{k x^{k-1}}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}}{(k-1)!}$$

(switch index)

$n = k-1$

If $k=1$, $n=0$

$k=0$: $\frac{0x^{-1}}{0!} = \frac{0}{1x} = 0$

First term $= 0$

ignore

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$$

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$f'(x) = 0 + 1 + \frac{\cancel{2}x}{\cancel{2}\cdot 1} + \frac{\cancel{3}x^2}{\cancel{3}\cdot 2\cdot 1} + \frac{\cancel{4}x^3}{\cancel{4}\cdot 3\cdot 2\cdot \cancel{1}} + \cdots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$= f(x)$$

$$f(x) = e^x$$

Final: 42/100 pts : short answer based on suggested book problems (X)

Historically: ~10/100 pts: Lagrange (7)

- <u>Don't</u> just check "corners"