Ch 9.1 : Approximating Functions with Polynomials

A polynomial is a <u>sum</u> of monomial terms

eg $x + 2x^5 + 7x^2 - 9$

Adding up simple <u>functions</u> - not just adding up #s.

ex Want a polynomial that behaves like $f(x) = \sin x$ near $\boxed{x=0}$

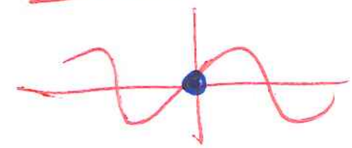
Say we want a cubic polynomial:

$$p(x) = ax^3 + bx^2 + cx + d$$

If $p(x) \approx f(x)$ near $x=0$:

- $p(0)=f(0)$:

$$\left.\begin{array}{l} f(0)=\sin 0=0 \\ p(0)=0+0+0+d \end{array}\right\} \quad \boxed{d=0}$$



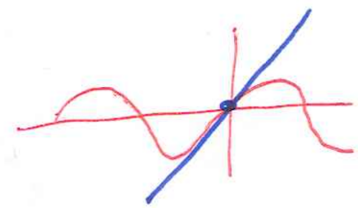
- $p'(0)=f'(0)$:

$f'(x)=\cos x$

$f'(0)=1$

$p'(x)=3ax^2+2bx+c$

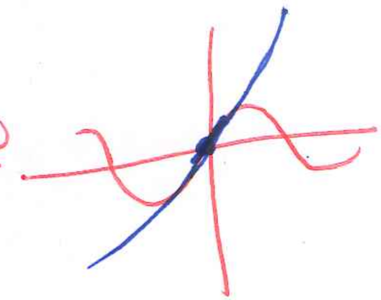
$p'(0)=c$ $\quad \boxed{c=1}$



- $p''(0)=f''(0)$:

$$\left.\begin{array}{l} f''(x)=-\sin x \\ f''(0)=-\sin 0=0 \end{array}\right] \quad 2b=0 \quad \boxed{b=0}$$

$p''(x)=6ax+2b \qquad p''(0)=2b$



- $p'''(0)=f'''(0)$

$f'''(x)=-\cos x$

$f'''(0)=-1$

$6a=-1$

$\boxed{a=-1/6}$

$p'''(0)=6a$

$p'''(x)=6a$

$f(x) = \sin x$

$p(x) = ax^3 + bx^2 + cx + d$

$= \frac{-1}{6}x^3 + 0 + 1\cancel{x} + 0$

$= x - \frac{x^3}{6}$

easy to evaluate (unlike sine)

cubic polynomial

@ $x=0$: matches $f(x) = \sin x$ in:

- value
- 1st deriv
- 2nd deriv
- 3rd deriv

Taylor Polynomial $T_N(x)$

approximating a function $f(x)$:

- choose a "centre" (two functions match)
- Make an N^{th} - degree polynomial (cubic, quadratic, etc)
- $T_N(a) = f(a)$ values match at centre
- $T_N'(a) = f'(a)$ 1st derivs match at centre
- $T_N''(a) = f''(a)$ etc

$\vdots$

- $T_N^{(N)}(a) = f^{(N)}(a)$ N^{th} derivatives match at centre

Formula:

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

centred at a

Differentiating monomials → factorials

eg

$f(x) = x^5$

$f'(x) = 5x^4$

$f''(x) = 5 \cdot 4 x^3$

$f'''(x) = 5 \cdot 4 \cdot 3 x^2$

$f^{(4)}(x) = 5 \cdot 4 \cdot 3 \cdot 2x$

$f^{(5)}(x)\ (5!)$

ex Find 100^{th}-degree Taylor polynomial for $f(x) = \cos x$, centre $a=0$.

Formula: $T_{100}(x) = \sum_{n=0}^{100} \frac{f^{(n)}(0)}{n!}(x-0)^n$

n: index, nothing to solve for

x: variable, " "

Need to find: $f^{(n)}(0)$ when $f(x) = \cos x$

Look for a Pattern

centre: $\boxed{0}$

$f(x) = \cos x$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$

$f^{(5)}(x) = -\sin x$

$f^{(6)}(x) = -\cos x$

$f^{(7)}(x) = \sin x$

$f^{(8)}(x) = \cos x$

$f(0) = 1$

$f'(0) = 0$

$f''(0) = -1$

$f'''(0) = 0$

$f^{(4)}(0) = 1$

$f^{(5)}(0) = 0$

$f^{(6)}(0) = -1$

$f^{(7)}(0) = 0$

$f^{(8)}(0) = 1$

$\vdots$

Odd derivs: all 0

Even derivs: ± 1

$$T_{100}(x) = \sum_{n=0}^{100} \frac{f^{(n)}(0)}{n!} x^n \quad \text{(formula, } a=0\text{)}$$

$\boxed{\text{Convention}}$ $0! = 1$

$$= \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \dots + \frac{f^{(100)}(0)}{100!}x^{100}$$

$$= \frac{1}{1!}1 + 0 + \frac{-1}{2!}x^2 + 0 + \frac{1}{4!}x^4 + \dots + \frac{1}{100!}x^{100}$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \frac{1}{10!}x^{10} + \dots + \frac{1}{100!}x^{100}$$

$$\boxed{\sum_{k=0}^{50} (-1)^k \frac{x^{(2k)}}{(2k)!}}$$

$f^{(0)}(0) = 1$

$f''(0) = -1$

$f''''(0) = 1$

$f^{(6)}(0) = -1$

div by 4: +1

(ex) Want to approx $e^x = f(x)$ centred at 0 with 20-degree Taylor polynomial.

Formula: $T_{20}(x) = \sum_{n=0}^{20} \frac{f^{(n)}(0)}{n!} x^n$

As always: need to find $f^{(n)}(0)$.

$f(x) = e^x$

$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = e^0 = 1$

So: $T_{20}(x) = \sum_{n=0}^{20} \frac{1}{n!} x^n$

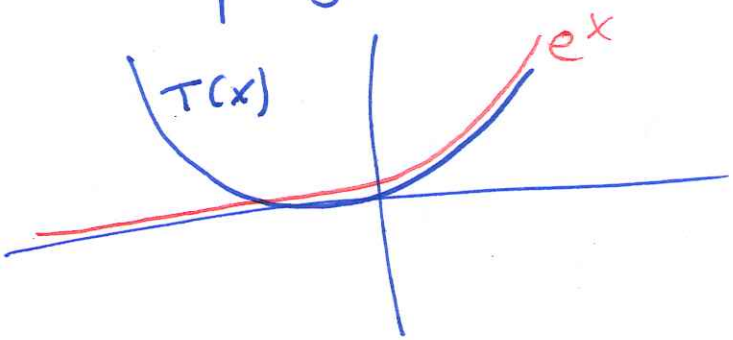
$= \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^{20}}{20!}$

Page 694 text: lots of examples

If we centre Taylor Series at 0:
we call it "Maclaurin Series"

(Ignore binomial $(1+x)^p$)

Q: How accurate are these approximating polynomials?



Taylor Remainder Let $T_n(x)$ be the Taylor polynomial of order n for $f(x)$.
The remainder in using $T_n(x)$ to approximate $f(x)$ at the point x is:

$$R_n(x) = \underbrace{f(x)}_{\text{actual}} - \underbrace{T_n(x)}_{\text{approx}}$$

Taylor's Remainder Theorem (p668)

Let f have continuous derivatives up to $f^{(n+1)}$ on an open interval I containing the centre, a.

For all x in I:

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

for some c between x & a.

(Computations similar to error is Simpson, Trap, MP) (ex) Suppose you want to approximate

$\cos(1)$

using a Taylor polynomial of $f(x) = \cos x$ centred at $a=0$. (Maclaurin)

What degree polynomial should you use to guarantee error ≤ 0.001

$$|R_n| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(x-0)^{n+1} \right|$$ for some c btw 1 and 0

x a

$f(x) = \cos x$

Derivs: $\mp \cos x$, $\mp \sin x$

Always: $|f^{(n+1)}(c)| \leq 1$

$$|R_n(1)| \leq \frac{1}{(n+1)!} 1^{n+1} = \frac{1}{(n+1)!} \leq 0.001$$

worst case scenario

find n makes this true (exam answer)

$$\frac{\pi^{n+1}}{(n+1)!} \leq \frac{1}{1000} = 0.001$$

$$\frac{\pi^{13}}{13!} \leq 0.001$$

$n+1 \geq 13$

$n \geq 12$

$n = 12$ suffices

deg-12 Taylor poly