[Announcement: Ch 9.3, 9.4 cut for time]

Remember from Last Time:

When faced with an infinite sum ("series") we should determine whether the sum converges or diverges. That is: as we add up more and more terms, is the sum nearing some fixed number? Or does it keep changing significantly?

?

<u>Theorem</u> (direct comparison test) ← known

Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n sufficiently large, then $\sum a_n$ is also convergent.

(ii) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n sufficiently large, then $\sum a_n$ is also divergent.

ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+\ln n} = \frac{1}{1} + \frac{1}{4+\ln 2} + \frac{1}{9+\ln 3} + \cdots$ Conv or Div?

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES (p-test, $p=2>1$)

$b_n = \frac{1}{n^2}$

$$\frac{1}{n^2+\ln} \leq \frac{1}{n^2}$$

$a_n = \frac{1}{n^2+\ln n}$

By comparison test,

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+\ln n}$ converges as well.

ex $\displaystyle\sum_{n=100}^{\infty} \frac{1}{2\sqrt{n}-9}$ Conv or Div?

Compare to $\displaystyle\sum \frac{1}{2\sqrt{n}} = \frac{1}{2}\sum \frac{1}{n^{1/2}}$ ($p=1/2$, DIVERGENT)

$$\frac{1}{2\sqrt{n}-9} > \frac{1}{2\sqrt{n}}$$ Can use direct comparison:

$$\sum \frac{1}{2\sqrt{n}-9}$$

ex $\displaystyle\sum \frac{1}{2\sqrt{n}+9}$ compare to $\frac{1}{2\sqrt{n}}$

$$\frac{1}{2\sqrt{n}+9} < \frac{1}{2\sqrt{n}}$$ can't use direct comparison

Limit Comparison Test:

Suppose a_n and b_n are sequences with only positive terms and $\lim_{n\to\infty} a_n/b_n$ is a real number other than 0.

Then $\sum a_n$ and $\sum b_n$ both conv or both div.

Want to compare unknown behaviour-series to a series whose behaviour is known.

If $\lim_{n\to\infty} \frac{a_n}{b_n}$ is real #, not 0, then comparison is OK.

Re-visit $\sum \frac{1}{2\sqrt{n}+9}$

Idea: compare to $\sum \frac{1}{2\sqrt{n}}$ (divergent by p-test)

To certify that the comparison is OK:

$$\lim_{n\to\infty} \frac{1}{2\sqrt{n}} \Big/ \frac{1}{2\sqrt{n}+9} = \lim_{n\to\infty} \frac{2\sqrt{n}+9}{2\sqrt{n}}$$

doesn't matter which goes on top

$$= \lim_{n\to\infty}\Big(1+\underbrace{\frac{9}{2\sqrt{n}}}_{\searrow 0}\Big) = 1$$

Real # Not 0

LCT tells us it's ok to compare two series - they both $\boxed{\text{diverge}}$

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ div if $p \le 1$

Sums (w/ pos terms):

CONV or DIV

"sum "small"" · sum "lg" $\to \infty$

smaller than a big thing:
could still be big
could be small
don't know

bigger than a big thing means it's big!

ex $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{n^4}$ Conv or Div?

Idea: compare to $\sum \frac{n^2}{n^4} = \sum \frac{1}{n^2}$

Note $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv $(p=2)$

$$\frac{n^2+1}{n^4} > \frac{n^2}{n^4} = \frac{1}{n^2}$$

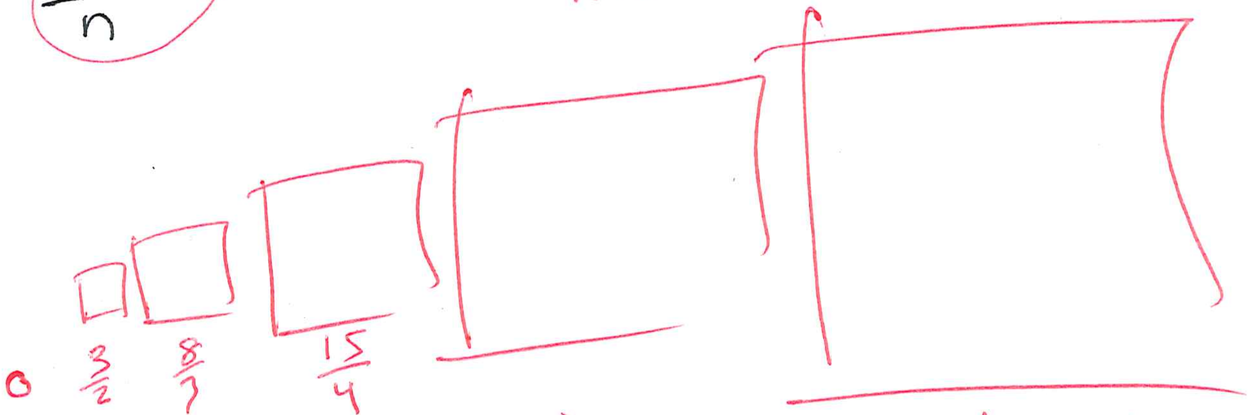
can't use direct comparison test

$$\lim_{n\to\infty} \frac{n^2+1}{n^4} \Big/ \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^2+1}{n^4}(n^2)$$

$$= \lim_{n\to\infty} \frac{n^2+1}{n^2} = \lim_{n\to\infty} \Big(1+\underbrace{\frac{1}{n^2}}_{0}\Big) = 1 \Big\} \begin{array}{l}\text{real number}\\ \neq 0\end{array}$$

LCT implies $\sum \frac{n^2+1}{n^4}$ converges as well.

ex $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n}$ $\qquad \displaystyle\lim_{n\to\infty} \frac{n^2-1}{n} = \infty$



b/c $\displaystyle\lim_{n\to\infty} \left(\frac{n^2-1}{n}\right) \neq 0$, series diverges

by divergence test.

ex $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^2}$ $\qquad \displaystyle\lim_{n\to\infty} \frac{n-1}{n^2} = 0$ (can't use div test)

idea: $\dfrac{n-1}{n^2} \approx \dfrac{n}{n^2} = \dfrac{1}{n}$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series)

diverges $(p=1)$

Need to prove this idea is ok

Note: $\frac{n-1}{n^2} < \frac{n}{n^2} = \frac{1}{n}$ "big"

Could use direct comp on $\sum \frac{n+1}{n^2}$

Can't use direct comp.

LCT: $\lim\limits_{n\to\infty} \frac{n-1}{n^2} \Big/ \frac{1}{n} = \lim\limits_{n\to\infty} \frac{n-1}{n^2} \cdot n$

$= \lim\limits_{n\to\infty} \left(\frac{n-1}{n}\right) = 1$ ← Real # not 0

LCT tells us $\sum \frac{n-1}{n^2}$ & $\sum \frac{1}{n}$ both conv/div

Since $\sum \frac{1}{n}$ div,

$\sum \frac{n-1}{n^2}$ $\boxed{\text{div}}$ as well.

Comparison tests require positive terms in sums.

Given a sequence a_n, create sequence $|a_n|$

eg $\underbrace{\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}}_{a_n} \to \underbrace{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}}_{|a_n|}$

Intuition:
to make a finite-valued series, it is helpful if terms are both + & -

should be "harder" for $\sum |a_n|$ to converge than for $\sum a_n$ to converge.

FACT:

if $\sum |a_n|$ conv then: $\sum a_n$ conv too

If $\sum |a_n|$ div then: $\sum a_n$ maybe conv, maybe div

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ex $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$ Conv or Div?

Terms: some +, some −

Consider instead: $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos n|}{n^2}$ ← positive terms

$$\frac{|\cos n|}{n^2} \le \frac{1}{n^2}$$

"small"

$\sum \frac{1}{n^2}$ conv ($p=2$)

$\cos x \le 1$ always

By direct comparison test,

$\sum \frac{|\cos n|}{n^2}$ conv

Therefore $\sum \frac{\cos n}{n^2}$ conv as well

If $\sum a_n$ conv and $\sum |a_n|$ conv
we say $\sum a_n$ converges absolutely

If $\sum a_n$ ~~div~~ conv and $\sum |a_n|$ div
we say $\sum a_n$ converges conditionally

If $\sum a_n$ div, then certainly $\sum |a_n|$ diverges
no special name

Ratio Test

$\sum a_n$ infinite series w/ positive terms
and define $r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

(1) If $0 \le r < 1$: series converges

(2) If $r > 1$: series diverges ∞

(3) If $r = 1$: need another test

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ex) $\sum_{k=1}^{\infty} \frac{k^2}{4^k}$ Conv or Div?

Initial Thoughts: $\frac{1}{4^k} = \left(\frac{1}{4}\right)^k$ geometric

$\frac{k^2}{4^k} \gg \frac{1}{4^k}$ can't use direct comparison

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{4^{n+1}} \Big/ \frac{n^2}{4^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{4^n}{4^{n+1}} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^2 \cdot \frac{1}{4} = \frac{1}{4} < 1$$

So $\sum \frac{k^2}{4^k}$ converges by ratio test.

ex $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\ln n)}{2^n}$ conv or div ?

Hint: $-1 \le \cos(\ln n) \le 1$

b/c of neg terms, most tests not available to us

Consider: $\displaystyle\sum_{n=1}^{\infty} \frac{|\cos(\ln n)|}{2^n}$

$$\frac{|\cos(\ln n)|}{2^n} \le \frac{1}{2^n}$$

$\sum \left(\frac{1}{2}\right)^n$ geometric, $r = 1/2$ so <u>converges</u>

Direct Comparison: $\displaystyle\sum \frac{|\cos(\ln n)|}{2^n}$ conv

So: $\displaystyle\sum \frac{\cos(\ln n)}{2^n}$ converges as well.

Factorials often do well under ratio test

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \ldots \cdot n$$

eg $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

products divide well!

$$\frac{7!}{6!} = \frac{\cancel{1} \cdot \cancel{2} \cdot \ldots \cdot \cancel{6} \cdot 7}{\cancel{1} \cdot \cancel{2} \cdot \ldots \cdot \cancel{6}} = 7$$

$$\frac{(n+1)!}{n!} = \frac{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \ldots \cdot \cancel{n} \cdot (n+1)}{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \ldots \cdot \cancel{n}} = n+1$$

ex $\sum_{n=1}^{\infty} \frac{n!\,n!}{(2n)!}$ Conv or Div ?

$n = k+1$
$\Rightarrow 2n = 2(k+1)$
$= 2k+2$

Ratio Test: $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} =$

$\lim_{k\to\infty} \frac{(k+1)!\,(k+1)!}{(2k+2)!} \Big/ \frac{k!\,k!}{(2k)!}$

$= \lim_{k\to\infty} \frac{(k+1)!}{k!} \cdot \frac{(k+1)!}{k!} \cdot \frac{(2k)!}{(2k+2)!}$

$= \lim_{k\to\infty} (k+1)(k+1) \cdot \frac{1}{(2k+1)(2k+2)}$

$= \lim_{k\to\infty} \frac{k^2+2k+1}{4k^2+6k+2} = \frac{1}{4} = r < 1$

By ratio test series converges.

$n!\,n! = (n!)^2 \neq (n^2)!$

$\frac{(k+1)!}{k!} : \frac{\cancel{1 \cdots k}(k+1)}{\cancel{1 \cdots k}}$

$\frac{(2k)!}{(2k+2)!} \quad \frac{\cancel{1 \cdot 2 \cdot 3 \cdots (2k)}}{\cancel{1 \cdot 2 \cdots 2k}(2k+1)(2k+2)}$

$(2k)! \neq 2(k!)$

eg $k=3$

$(2k)! = 6! = 6\cdot5\cdot4\cdot3\cdot2\cdot1$

$2(k!) = 2(3!) = 2\cdot3\cdot2\cdot1$

Can't factor const out of factorials!

Telescoping Series

ex $\sum_{n=1}^{\infty}\left(\frac{1}{n+2}-\frac{1}{n+3}\right) = \lim_{N\to\infty} S_N = \frac{1}{3}$

converges to $1/3$

$n=1$ $\quad \frac{1}{3} - \cancel{\frac{1}{4}}$

$n=2$ $\quad \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}$

$n=3$ $\quad \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}}$

$n=4$ $\quad \cancel{\frac{1}{6}} - \frac{1}{7}$

$\vdots$

$s_1 = \frac{1}{3} - \frac{1}{4}$

$s_2 = \frac{1}{3} - \frac{1}{5}$

$s_3 = \frac{1}{3} - \frac{1}{6}$

$s_4 = \frac{1}{3} - \frac{1}{7}$

$S_N = \frac{1}{3} - \frac{1}{N+3}$

ex $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}-\frac{n+1}{n+2}\right) = \lim_{N\to\infty}\left(\frac{1}{2}-\frac{N+1}{N+2}\right) = \frac{1}{2}-1 = \frac{-1}{2}$

(conv)

$n=1: \quad \frac{1}{2} - \cancel{\frac{2}{3}}$

$n=2: \quad \cancel{\frac{2}{3}} - \cancel{\frac{3}{4}}$

$n=3: \quad \cancel{\frac{3}{4}} - \cancel{\frac{4}{5}}$

$n=4: \quad \cancel{\frac{4}{5}} - \frac{5}{6}$

$$S_N = \frac{1}{2} - \frac{N+1}{N+2}$$

ex $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ DIV

Note: $\lim_{n\to\infty} \frac{n+1}{n} = 1$ and $\ln(1)=0$

Note: $\ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n$

$\ln(A/B) = \ln A - \ln B$

$n=1 \quad \cancel{\ln 2} - \cancel{\ln 1}^{\,0}$

$n=2 \quad \cancel{\ln 3} - \cancel{\ln 2}$

$n=3 \quad \cancel{\ln 4} - \cancel{\ln 3}$

$n=4 \quad \cancel{\ln 5} - \cancel{\ln 4}$

$n=5 \quad \ln 6 - \cancel{\ln 5}$

$\ln 2 = S_1$

$S_2 = \ln 3$

$S_3 = \ln 4$

$S_4 = \ln 5$

$S_N = \ln(N+1) \xrightarrow[N\to\infty]{} \infty$