Infinite Series:

$$\sum_{n=a}^{\infty} a_n = \lim_{N \to \infty} \underbrace{\sum_{n=a}^{N} a_n}_{\text{partial sums}} = \lim_{N \to \infty} S_N$$

Limit exists: "converges"

Limit DNE (including $\pm\infty$) "diverges"

Divergence Test: If $\lim_{n \to \infty} a_n \neq 0$

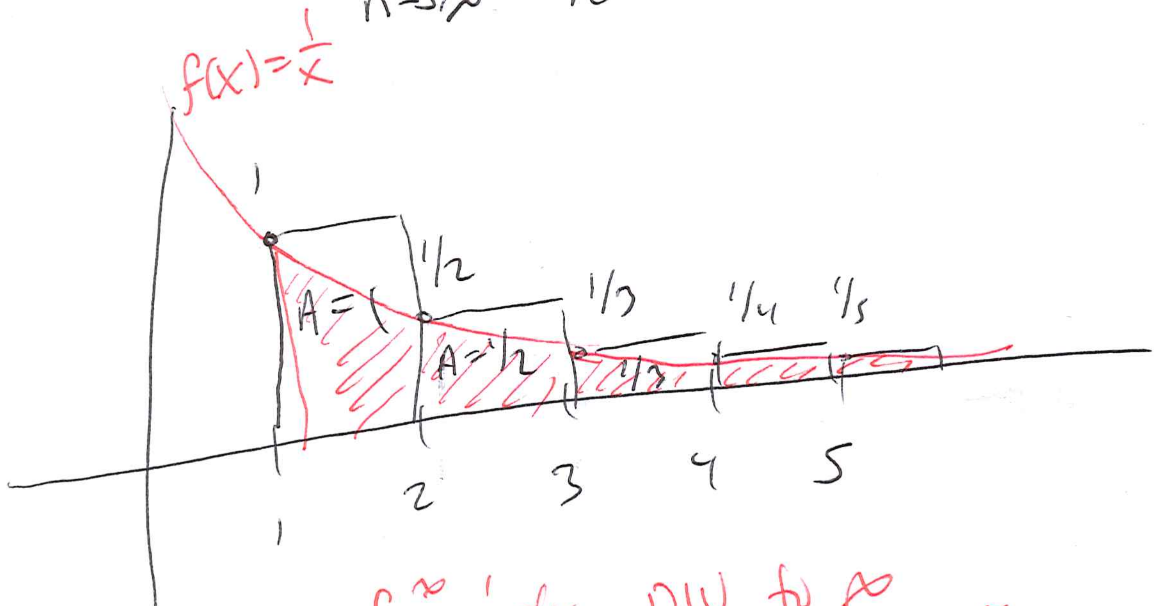
then $\sum_{n=a}^{\infty} a_n$ DIVERGES

If $\lim_{n\to\infty} a_n = 0$: could be $\sum a_n$ div, or conv

need more tests.

ex "harmonic series": $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

$\lim_{n\to\infty} \frac{1}{n} = 0$ (Div Test doesn't apply)



$\sum_{n=1}^{\infty} \frac{1}{n}$: total area under rectangles

$\int_1^{\infty} \frac{1}{x}\,dx \leq \sum_{n=1}^{\infty} \frac{1}{n}$

But: $\int_1^{\infty} \frac{1}{x}\,dx$ DIV to ∞

We conclude: $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES as well

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{n\to\infty} \int_1^n \frac{1}{x}\,dx = \lim_{n\to\infty} \left(\ln|x| \Big|_1^n \right)$$

$$= \lim_{n\to\infty} \left(\ln n - \cancel{\ln 1} \right) = \infty$$

Integral Test:

Suppose f is continuous, positive, decreasing on $[a,\infty)$

Let $a_n = f(n)$. Then:

$\sum_{n=a}^{\infty} a_n$ conv/div if and only if

$\int_a^{\infty} f(x)dx$ conv/div

ex $\sum_{n=10}^{\infty} \frac{1}{n \ln n}$ Conv or Div?

Div test does not apply:

$\lim_{n\to\infty} \frac{1}{n \ln n} = 0$

$f(x) = \frac{1}{x \ln x}$ on $[10, \infty)$:

continuous, positive, decreasing

Integral test applies:

$$\int_{10}^{\infty} \frac{1}{x \ln x}\, dx = \int_{\ln 10}^{\infty} \frac{1}{u}\, du$$

DIVERGENT

$u = \ln x$

$du = \frac{1}{x} dx$

p-test ($p=1$)

So by integral test, also

$$\sum_{n=10}^{\infty} \frac{1}{n \ln n}$$

DIVERGES.

$$\frac{1}{10 \ln 10} + \frac{1}{11 \ln 11} + \frac{1}{12 \ln 12} + \cdots$$

sum grows without bound

Recall p-test for integrals:

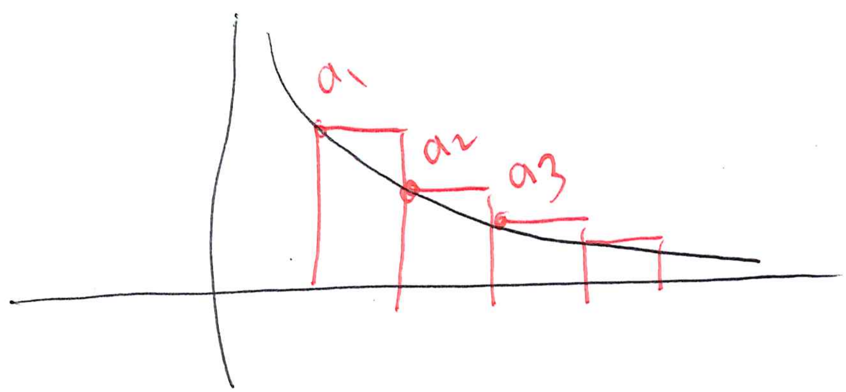
$$\int_1^\infty \frac{1}{x^p}\,dx : \begin{array}{ll} \underline{\text{conv}} & \text{if } p>1 \\ \underline{\text{DIV}} & \text{if } p\le 1 \end{array}$$

Also have p-test for series:

$a_n = \frac{1}{n^p}$ (constant p) : $f(x) = \frac{1}{x^p}$ - positive - cont - dec on $(1,\infty)$

so by integral test:

$$\sum_{n=1}^\infty \frac{1}{n^p} \text{ conv / div as } \int_1^\infty \frac{1}{x^p}\,dx \text{ conv/div}$$



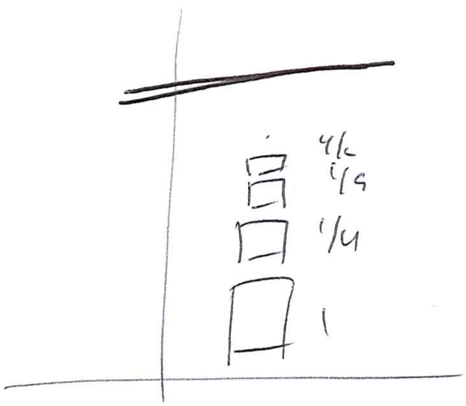
p-test for series:

$$\sum_{n=a}^{\infty} \frac{1}{n^p} : \begin{cases} \underline{\text{conv}} & \text{if } p>1 \\ \underline{\text{div}} & \text{if } p \le 1 \end{cases}$$

ex (Harmonic) $\sum \frac{1}{n}$ $p=1$ DIVERGENT SERIES

ex $$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots$$

$p=2$ so series <u>converges</u>



sum gets closer & closer to some fixed number

Ch 8.5|

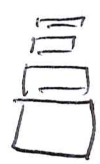
ex $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ <u>converges</u> (p-test)

What about $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n+2}$?

Could use $\int$ test: partial frac or trig-sub
looks undesirable

Intuition: should behave similarly to $\sum \frac{1}{n^2}$
b/c for large n, n^2+n+2 dominated by n^2

$$\frac{1}{n^2+n+2} < \frac{1}{n^2}$$



Adding up $\left(\frac{1}{n^2}\right)$'s gives <u>finite</u> number (NOT $\to \infty$)

So: adding up $\left(\frac{1}{n^2+n+2}\right)$'s
<u>also</u> gives <u>finite</u> number
(even smaller)

(Direct) Comparison Test

Let a_n, b_n be series with positive terms.

know about

curious about

curious

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for n sufficiently large then $\sum a_n$ converges as well.

curious

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$, then $\sum a_n$ diverges as well.

If $\sum_{n=1}^{\infty} a_n$ conv/div, then also $\sum_{n=1000000}^{\infty} a_n$ conv/div

finite difference

ex $\sum \frac{1}{\sqrt{n}-1}$: conv or div?

Intuition: $\sum \frac{1}{\sqrt{n}-1}$ should behave similarly to $\sum \frac{1}{\sqrt{n}}$

Need to show intuition is justified.

Let $a_n = \frac{1}{\sqrt{n}-1}$, $b_n = \frac{1}{\sqrt{n}}$

- $\sum b_n = \sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ <u>DIVERGES</u> by p-test

  p-series, $p = 1/2$

- $\sqrt{n} - 1 < \sqrt{n}$, so $\frac{1}{\sqrt{n}-1} > \frac{1}{\sqrt{n}}$

  so $a_n > b_n$

- By comparison test, also $\sum \frac{1}{\sqrt{n}-1}$ <u>DIVERGES</u>

Note: a_n, b_n <u>positive</u>

ex $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$; conv or div?

Intuition: for big n, $n^2+5 \approx n^2$

<u>Need to justify</u>

Let $a_n = \frac{1}{n^2+5}$, $b_n = \frac{1}{n^2}$

- both have <u>positive</u> terms
- $\sum \frac{1}{n^2}$ <u>conv</u> $(p=2)$
- $n^2+5 > n^2$ so $\frac{1}{n^2+5} < \frac{1}{n^2}$, so $a_n < b_n$

So by comparison test, also $\sum \frac{1}{n^2+5}$ <u>CONVERGE</u>

ex $\sum_{n=1}^{\infty} \frac{1+n}{n^3}$ conv or div?

Intuition: for large n,

$$\frac{1+n}{n^3} \approx \frac{n}{n^3} = \frac{1}{n^2}$$

Need to justify intuition:

$a_n = \frac{1+n}{n^3}$ $\qquad$ $b_n = \frac{1}{n^2}$

$\sum b_n$ CONV $(p=2)$

But $a_n = \frac{1+n}{n^3} > \frac{n}{n^3} = \frac{1}{n^2} = b_n$

Can't use direct comparison test

→ to justify intuition

Limit Comparison Test to the rescue
(Thursday)