Recall from Last Time:

A <u>series</u> is an infinite sum. We define it as a limit of partial sums:

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} \underbrace{\sum_{n=1}^{N} a_n}_{\text{sum of } a_1 \text{ through } a_N} = \lim_{N\to\infty} S_N$$

If the limit exists (that is, it's a real number) we say the series <u>converges</u>. If the limit does not exist (including $\pm\infty$) we say the series <u>diverges</u>.

Integral Test: Let $f(x)$ be positive, decreasing, and continuous on $[a,\infty)$, and let $a_n = f(n)$ for integers $n \geq a$. Then $\sum_{n=a}^{\infty} a_n$ converges if and only if $\int_a^{\infty} f(x)\,dx$ converges.

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Recall: $\int_a^\infty \frac{1}{x^p}\,dx \qquad (a>0)$

$f(x) = \frac{1}{x^p}$: positive continuous decreasing when $x>0$

if $p>1$: converges
if $p \le 1$: diverges

By integral test: $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$

if $p>1$: converges
if $p \le 1$: diverge

eg $\sum_{n=1}^{\infty} \frac{1}{n}$: $p=1$ so diverges by p-test

eg $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \dots$ $p=2$, so converges by p-test

Ch 8.5 Ratio & Comparison Tests

Comparison, Limit Comparison: tests to show $\sum_{n=a}^{\infty} a_n$ conv or div.

(ex) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$ conv or div?

Divergence test is easiest, but doesn't always apply

Adding up $\frac{1}{\sqrt{n}-1} \xrightarrow{n\to\infty} 0$

Inconclusive - div test doesn't help

Not easy to integrate - don't want to use integral test.

Kind of looks like $\frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$

$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ DIVERGES by p-test ($p = 1/2$)

Note:

$$\sqrt{n} - 1 < \sqrt{n}$$

so $\dfrac{1}{\sqrt{n}-1} > \dfrac{1}{\sqrt{n}}$

Adding up these terms gives ever-increasing sum

adding up even bigger terms

So $\sum_{2}^{\infty} \dfrac{1}{\sqrt{n}-1}$ also diverges

(Direct) Comparison Test

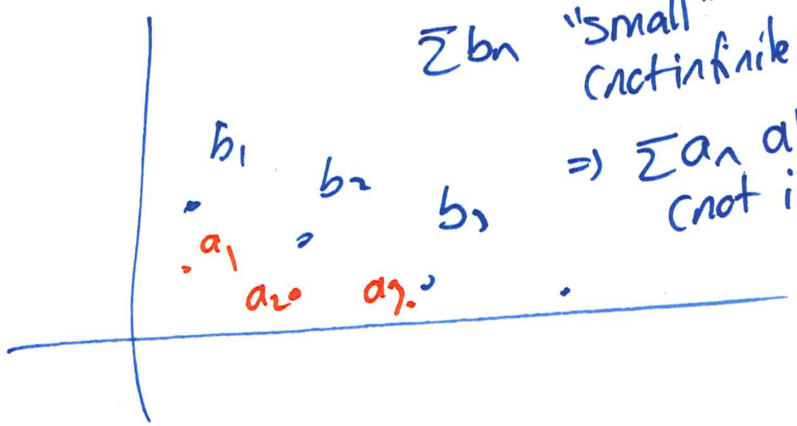
Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $a_n \le b_n$ for all n sufficiently large, then $\sum a_n$ converges as well.

(ii) If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n sufficiently large, then $\sum a_n$ ~~converges~~ diverges as well.

$\sum b_n$ "small" (not infinite)

$\Rightarrow \sum a_n$ also "small" (not infinite)



(i)

(ex) $\sum_{n=1}^{\infty} \frac{1}{n^2+5}$ conv or div?

Div test doesn't apply b/c $\frac{1}{n^2+5} \to 0$

$\sum \frac{1}{n^2}$ conv by p-test ($p = 2 > 1$)

$\frac{1}{n^2+5} < \frac{1}{n^2}$

By comparison test, $\sum \frac{1}{n^2+5}$ converges.

(ex) $\sum_{n=3}^{\infty} \frac{\arctan n}{\ln n}$

Notice: $\arctan 3 \le \arctan n \le \pi/2$ for $n \ge 3$

For large n, $\ln n < n$

don't know

$$\boxed{\frac{\arctan n}{\ln n} > \frac{\arctan 3}{n}}$$

$$\sum \frac{\arctan 3}{n} = (\arctan 3) \underbrace{\boxed{\sum \frac{1}{n}}}_{\text{DIV by p-test}}$$

know

"harmonic series"

So by comparison test, $\sum \frac{\arctan n}{\ln n}$ diverges

Note: if $\sum_{n=1}^{\infty} a_n$ conv, (div) then

$\sum_{n=100}^{\infty} a_n$ conv (div)

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_{100} + a_{101} + a_{102} + \cdots$$

$$\sum_{n=100}^{\infty} a_n = a_{100} + a_{101} + a_{102} + \cdots$$

$\underbrace{}_{\text{finite }\#}$

$a_1 + \cdots + a_{99}$

ex $\sum_{n=1}^{\infty} \frac{1+n}{n^3}$ conv or div?

Check: $\frac{1+n}{n^3} \longrightarrow 0$

Div test not useful.

Idea: $\frac{1+n}{n^3} \approx \frac{1}{n^2}$

$\sum \frac{1}{n^2}$ converges $(p=2)$

$\frac{1+n}{n^3} > \frac{n}{n^3} = \frac{1}{n^2}$ can't use direct comparison test

Limit Comparison Test: Suppose a_n and b_n are sequences with positive terms and $\lim_{n\to\infty} \frac{a_n}{b_n}$ is a real number greater than 0, then $\sum a_n$ & $\sum b_n$ both conv or both div

when you want to compare two series this "certifies" that the comparison is valid.

Let $a_n = \frac{1+n}{n^3}$, $b_n = \frac{1}{n^2}$

Evaluate $\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \left(\frac{1+n}{n^3}\right) / \left(\frac{1}{n^2}\right)$

$= \lim_{n\to\infty} \frac{n^2+n^3}{n^3} = \lim_{n\to\infty} \frac{n^2}{n^3}+1$

$= \lim_{n\to\infty} \left(\frac{1}{n}+1\right) = 1$ (real # >0)

Because (p-test) $\sum \frac{1}{n^2}$ converges, ($p=2$)
the limit comparison test tells us
$\sum \frac{1+n}{n^3}$ converges as well.

Limit comparison used when $\leq$ for direct
comparison doesn't work.

ex $\displaystyle\sum_{n=2}^{\infty} \frac{n^2-1}{n}$ Conv or Div?

$$\lim_{n\to\infty} \frac{n^3-1}{n} = \infty \neq 0$$

So, by divergence test, $\boxed{\text{diverges}}$

ex $\displaystyle\sum_{n=2}^{\infty} \frac{n-1}{n^2}$ Conv or div?

$$\lim_{n\to\infty} \frac{n-1}{n^2} = 0$$

Div test not helpful

Idea: $\displaystyle\frac{n-1}{n^2} \approx \frac{n}{n^2} = \frac{1}{n}$

$\displaystyle\sum \frac{1}{n}$ $\boxed{\text{DIV}}$ (harmonic series, $p=1$)

For direct comparison to work, would need

$$\frac{n-1}{n^2} \geq \frac{1}{n} \quad \text{(FALSE)}$$

can't use direct comparison

Use: limit comparison

Evaluate $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(\frac{n-1}{n^2}\right) \Big/ \left(\frac{1}{n}\right)$

$$= \lim_{n\to\infty} \underbrace{\frac{n^2-n}{n^2}}_{\frac{n^2}{n^2}-\frac{n}{n^2}} = \lim_{n\to\infty} \left(1-\frac{1}{n}\right) = \underline{1}$$

real #
not 0

That means: Lim Comp Test tells us our comparison is valid:
$\sum \frac{1}{n}$ and $\sum \frac{n-1}{n^2}$
both conv or both div

By lim comp test,
$\sum \frac{n-1}{n^2}$ ~~converge~~ diverges as well.

Ratio Test

Let $\sum a_k$ be an infinite series with positive terms and define

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

1. If $0 \le r < 1$, $\sum a_k$ converges
2. If $r > 1$ (or $r = \infty$) the series diverges
3. If $r = 1$, need another test

(ex) $\sum \frac{k^2}{4^k}$ conv or div?

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^2}{4^{n+1}} \Big/ \frac{n^2}{4^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{n^2} \cdot \frac{4^n}{4^{n+1}} = \lim_{n\to\infty} \underbrace{\left(\frac{n+1}{n}\right)^2}_{\to 1} \cdot \frac{1}{4} = \frac{1}{4}$$

Since $r < 1$, $\sum \frac{k^2}{4^k}$ converges.

Ratio test usually first choice for <u>factorials</u>

Recall: $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

eg $5! = 120$, because $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$

eg $\dfrac{(n+1)!}{n!} = \dfrac{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \ldots \cdot \cancel{(n)}(n+1)}{\cancel{1} \cdot \cancel{2} \cdot \cancel{3} \cdot \ldots \cdot \cancel{n}} = n+1$

Factorials + Division = ♥

ex $\displaystyle\sum_{n=1}^{\infty} \frac{n!\,n!}{(2n)!}$ conv or div?

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!\,(n+1)!}{[2(n+1)]!} \bigg/ \frac{n!\,n!}{(2n)!}$$

$$= \lim_{n\to\infty} \frac{(n+1)!\,(n+1)!}{n!\,n!} \cdot \frac{(2n)!}{(2n+2)!}$$

$$\lim_{n\to\infty} (n+1)(n+1) \cdot \frac{\cancel{1}\cdot\cancel{2}\cdot\cancel{\cdot}\cdot\ (2\cancel{n})}{\cancel{1}\cdot\cancel{2}\cdot\cancel{3}\cdots\cdot\cancel{(2n)}(2n+1)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{(n+1)^2}{(2n+1)(2n+2)} = \frac{1}{4} < 1$$

So by ratio test: <u>convergent</u> series.

rational:

$$\lim_{n\to\infty} \frac{n^2+2n+1}{4n^2+6n+2} = \frac{1}{4}$$