Recall from Last Time:

A <u>sequence</u> is a list of numbers. One kind of sequence is a <u>geometric</u> sequence. These have a common ratio between terms: $\{a_n\} = \{a_0 r^n\}$.

$a_0, a_0 r, a_0 r^2, a_0 r^3, a_0 r^4, \ldots$

We create a <u>sequence of partial sums</u> of $\{a_n\}$ by defining $S_N = \sum_{k=1}^{N} a_k = a_1 + a_2 + \cdots + a_N$.

eg

Sequence, a_n	Sequence of Partial Sums S_N	
$a_1 = 2$	$S_1 = 2$	just a_1
$a_2 = 4$	$S_2 = 6$	combine a_1, a_2
$a_3 = 6$	$S_3 = 12$	add a_1, a_2, a_3
$a_4 = 4$	$S_4 = 16$	add a_1, a_2, a_3, a_4
$a_5 = 2$	$S_5 = 18$	add a_1, a_2, a_3, a_4, a_5

A geometric partial sum

$$S_N = \sum_{k=0}^{N} ar^k = a\frac{1-r^{N+1}}{1-r}$$

$a, ar, ar^2, \dots, ar^N$

ex) Evaluate $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$

$(\frac{1}{3})^0 \quad (\frac{1}{3})^1 \quad (\frac{1}{3})^2 \quad (\frac{1}{3})^4$

$a = 1$

$r = 1/3$

$$= \sum_{k=0}^{4} \left(\frac{1}{3}\right)^k = \boxed{\frac{1-(1/3)^5}{1-1/3}}$$

ex) Evaluate $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81}$

Would like it if first term were 1

$$= \frac{2}{3}\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}\right)$$

$$= \frac{2}{3}\sum_{k=0}^{3}\left(\frac{1}{3}\right)^k = \boxed{\frac{2}{3}\left(\frac{1-(1/3)^4}{1-1/3}\right)}$$

Ch 8.3 Infinite Series

$$\underbrace{\sum_{k=a}^{\infty} a_k}_{\text{Adding up inf. many numbers – how???}} = \lim_{N\to\infty} \underbrace{\sum_{k=a}^{N} a_k}_{\text{Finite sum – no problem Definitely exists!}}$$

Geometric Series:

$$\sum_{n=0}^{\infty} r^n = \lim_{N\to\infty} \underbrace{\sum_{n=0}^{N} r^n}_{\text{formula}} = \lim_{N\to\infty} \frac{1-r^{N+1}}{1-r}$$

$$= \begin{cases} \text{DIV} & \text{if } |r|>1 \quad (r^{N+1} \to \pm\infty) \\ \dfrac{1}{1-r} & \text{if } |r|<1 \quad (r^{N+1} \to 0) \end{cases}$$

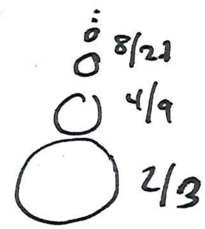
(ex) evaluate $\displaystyle\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^k$

$r = \frac{2}{3}$

$\left|\frac{2}{3}\right| < 1$

formula $\boxed{\dfrac{1}{1-2/3}}$

$\frac{1}{1-2/3}$



(ex) evaluate $\displaystyle\sum_{k=0}^{\infty}\left(-\frac{10}{9}\right)^k$

$r = -\frac{10}{9}$

$|r| > 1$

$\boxed{\text{DIV}}$

(ex) evaluate $\displaystyle\sum_{k=2}^{\infty}\frac{10^k}{9^{2k}}$

$(?)^k$

$$= \sum_{k=2}^{\infty}\frac{10^k}{81^k} = \sum_{k=2}^{\infty}\left(\frac{10}{81}\right)^k$$

$r = \frac{10}{81}$ $\quad |r| < 1$

Σ will conv

$$\sum_{k=2}^{\infty}\left(\frac{10}{81}\right)^k = \left(\frac{10}{81}\right)^2 + \left(\frac{10}{81}\right)^3 + \left(\frac{10}{81}\right)^4 + \left(\frac{10}{81}\right)^5 + \cdots$$

want to be 1 : factor

$$\left(\frac{10}{81}\right)^2\left[1 + \frac{10}{81} + \left(\frac{10}{81}\right)^2 + \left(\frac{10}{81}\right)^3 + \cdots\right]$$

$$\frac{10^2}{81^2}\sum_{k=0}^{\infty}\left(\frac{10}{81}\right)^k \underset{\text{formula}}{=} \boxed{\frac{10^2}{81^2}\cdot\frac{1}{1-10/81}}$$

"convergent"

ex

$$\sum_{k=5}^{\infty}\frac{3^{k+1}}{2^{2k}} = \sum_{k=5}^{\infty}\frac{3\cdot 3^k}{4^k} = 3\sum_{k=5}^{\infty}\left(\frac{3}{4}\right)^k$$

$2^{2n} = (2^2)^n = 4^n$

$r = \frac{3}{4}$

$|r| < 1$

will conv

let $n = k-5$

- $k = n+5$
- if $k=5$, $n=0$

change index

$$= 3\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^{n+5} = 3\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\left(\frac{3}{4}\right)^5$$

$$= 3\left(\frac{3}{4}\right)^5\underbrace{\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n}_{\text{formula}} = \boxed{\frac{3^6}{4^5}\cdot\frac{1}{1-3/4}}$$

Ch 8.4 : Divergence & Integral Tests

$$\sum_{n=1}^{\infty} 1 = 1+1+1+1+\cdots = \infty$$ DIVERGENT SERIES

$$\sum_{n=1}^{\infty} \left(\frac{1}{10} + \frac{1}{2^n}\right) = \frac{1}{10} + \frac{1}{2} + \frac{1}{10} + \frac{1}{4} + \frac{1}{10} + \frac{1}{8} + \frac{1}{10} + \frac{1}{16} \cdots = \infty$$ DIV

$$\sum_{n=1}^{\infty} (-1)^n = \lim_{N\to\infty} \sum_{n=1}^{N} (-1)^n = \lim_{N\to\infty} S_N \text{ DNE}$$

So: Series $\underline{\underline{\text{DIV}}}$

$-1 + 1 - 1 + 1 - 1 + 1 - 1 + 1$

$(-1+1) + (-1+1) + (-1+1) + \cdots = 0 + 0 + 0 = \underline{\underline{0}}$

$-1 + (1-1) + (1-1) + (1-1) + \cdots = -1 + 0 + 0 + 0 \cdots = \underline{\underline{-1}}$

Need to use <u>definition</u> of series (not intuition)

Partial sums:

$-1, 0, -1, 0, -1, 0, -1, 0$ limit DNE

Divergence test:

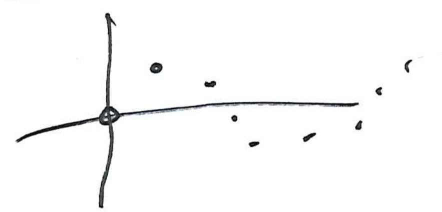
If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum\limits_{n=a}^{\infty} a_n$ DIVERGES

$$\sum_{n=a}^{\infty} a_n$$

$\lim\limits_{n\to\infty} a_n \neq 0$ → series DIV

$\lim\limits_{n\to\infty} a_n = 0$ → could conv, could div
we'll learn more tests

ex $\sum\limits_{n=1}^{\infty} \sin n$ $\lim\limits_{n\to\infty} \sin n$ DNE



So, by DIV TEST,

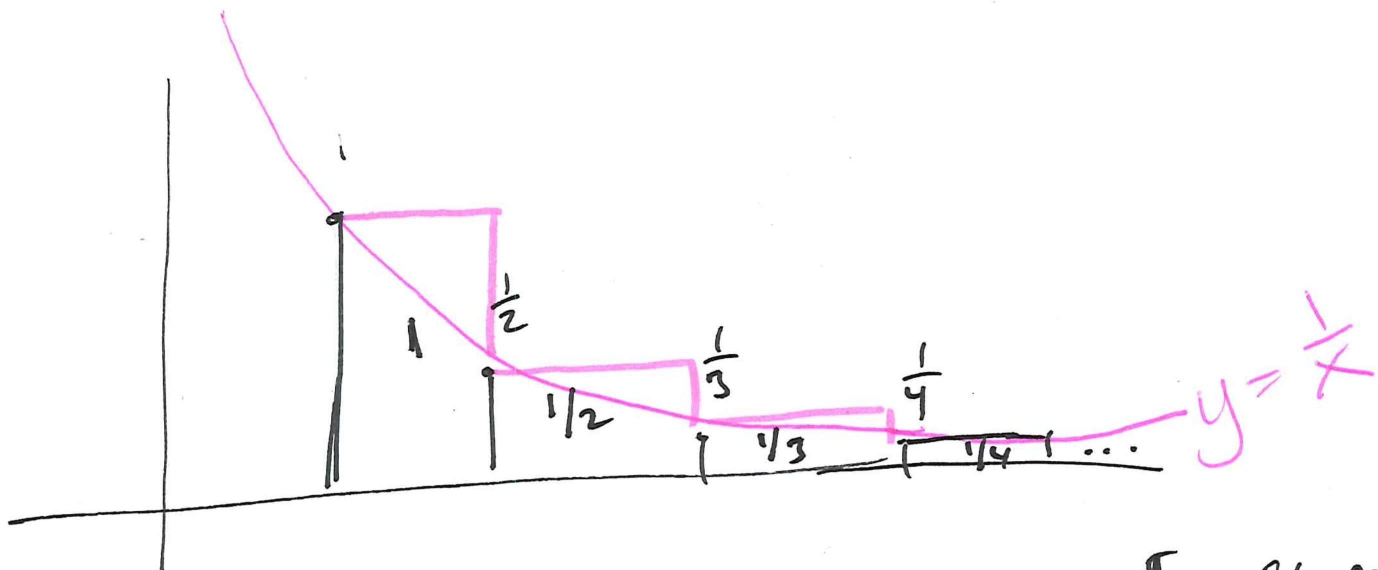
$\sum\limits_{n=1}^{\infty} \sin n$ DIVERGES ex $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$

"harmonic series"

$\lim_{n\to\infty} \frac{1}{n} = 0$

Div Test tells us nothing

Need to look further



$\sum_{n=1}^{\infty} \frac{1}{n}$: sum of areas of rectangles ∮ (base = 1)

$\int_1^{\infty} \frac{1}{x}\,dx$: area under curve $\frac{1}{x}$

See: $\underbrace{\int_1^{\infty} \frac{1}{x}\,dx}_{\infty} \leq \sum_{n=1}^{\infty} \frac{1}{n}$

So: $\sum_{n=1}^{\infty} \frac{1}{n}$ $\boxed{\text{DIV}}$

Integral Test:

Suppose f is a continuous, positive, decreasing function on $[a, \infty]$ and let $a_n = f(n)$. Then the series $\sum_{n=a}^{\infty} a_n$ is convergent / divergent if and only if $\int_a^{\infty} f(x)dx$ is conv/div.

ex Does $\sum_{2}^{\infty} \frac{1}{n \ln n}$ converge or diverge?

$f(x) = \frac{1}{x \ln x}$

continuous on $[2, \infty)$

positive "

decreasing "

$\int_2^{\infty} \frac{1}{x \ln x} dx = \int_{\ln 2}^{\infty} \frac{1}{u} du$ DIV (by p-test)

$u = \ln x$

$du = \frac{1}{x} dx$

$p=1$

So, by integral test $\sum_{2}^{\infty} \frac{1}{n \ln n}$ DIV as well (ex) Does $\sum_{n=5}^{\infty} \frac{1}{n^2+1}$ conv or div?

$f(x) = \frac{1}{x^2+1}$ continuous, positive, dec on $[5, \infty)$ } can apply $\int$ test

$$\int_5^{\infty} \frac{1}{x^2+1}\,dx = \lim_{N \to \infty} \int_5^N \frac{1}{x^2+1}\,dx$$

$$= \lim_{N \to \infty} (\arctan N - \arctan 5) = \underbrace{\pi/2 - \arctan(5)}_{\text{real } \#}$$

$\int$ conv, so by $\int$ test,

$\sum_{n=5}^{\infty} \frac{1}{n^2+1}$ converges as well. (Don't know what to)