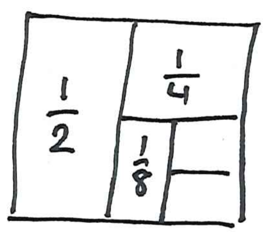
Remember from Last Time

A sequence is a list of numbers.

A series is the sum of the terms in a sequence.

e.g.



Sequence: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

Its limit is $\boxed{0}$

Series: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

Its limit is $\boxed{1}$.

Sequence of Partial Sums

a_1	$S_1 = a_1$	
a_2	$S_2 = a_1 + a_2$	sum of a_1, a_2
a_3	$S_3 = a_1 + a_2 + a_3$	" " first three terms
a_4	$S_4 = a_1 + a_2 + a_3 + a_4$	" " " 4 "
$\vdots$		

sequence — sequence of partial sums

ex: $\{a_n\}: 1, 1, 1, 1, 1, 1, 1, \ldots$ $\quad \lim_{n\to\infty} a_n = 1$

partial sums $\{S_n\}: 1, 2, 3, 4, 5, \ldots$ $\quad \lim_{n\to\infty} S_n = \infty$

ex ~~Series~~ sequence $\{a_n\}$

partial sums $\{S_n\}$

$a_1 = \frac{1}{2}$

$a_2 = \frac{1}{4}$

$a_3 = \frac{1}{8}$

$a_4 = \frac{1}{16}$

$a_5 = \frac{1}{32}$

$\lim_{n\to\infty} a_n = 0$

↑

size of tiles getting feensy

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{4}{8} + \frac{2}{8} + \frac{1}{8} = \frac{7}{8}$

$S_4 = 1 - \frac{1}{16} = \frac{15}{16}$

$S_5 = 1 - \frac{1}{32} = \frac{31}{32}$

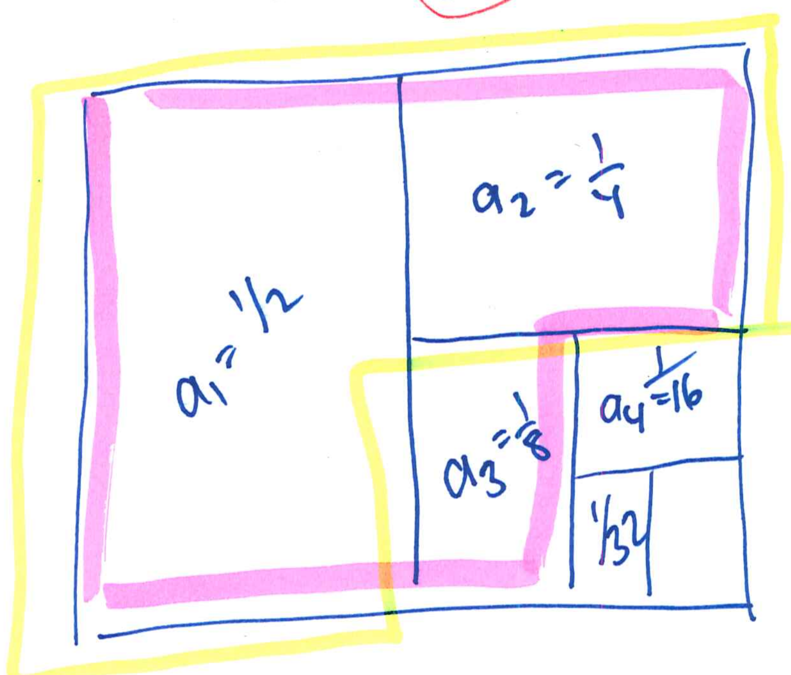
$\lim_{n\to\infty} S_n = 1$

↑

sum of first n ~~t~~ tile sizes

$a_n = \frac{1}{2^n}$

$S_n = 1 - \frac{1}{2^n}$



$S_2 = 1 - \frac{1}{4}$

$S_3 = 1 - \frac{1}{8}$

1

ex

$$S_4 - S_3 = \begin{aligned}&(a_1 + a_2 + a_3 + a_4)\\ -&(a_1 + a_2 + a_3)\end{aligned} = a_4 = \frac{1}{16}$$

$$S_{15} - S_{14} = \begin{aligned}&(a_1 + \cdots + a_{14} + a_{15})\\ -&(a_1 + \cdots + a_{14})\end{aligned} = a_{15} = \frac{1}{2^{15}}$$

ex

$\{S_N\}_1^\infty$: seq of partial sums of some sequence $\{a_n\}_1^\infty$

$S_N = N^2 + 1$ eg $S_1 = 2 = a_1$, $S_2 = 5 = a_1 + a_2$ $\Big\}$ $a_2 = 3$ $(2 \times 2 - 1 = 3)$

$S_3 = 10$

$\vdots$

- What is a_2?
- What is a_{100}? 199
- What is a_n?

$$\underbrace{S_{100}}_{\text{collect first 100 terms}} - \underbrace{S_{99}}_{\text{getrid of first 99 terms}} = a_{100}$$

$(2(100) - 1 = 199)$

$(100^2 + 1) - (99^2 + 1)$

$= 100^2 - 99^2 = (100 + 99)(\cancel{100 - 99}) = 199$

$$\underbrace{S_n - S_{n-1}}_{} = a_n$$

$(n^2 + 1) - ((n-1)^2 + 1) =$

$n^2 - (n-1)^2 = (n + n - 1)(\cancel{n - (n-1)})$

$= 2n - 1$

Geometric Sums

Geometric series: common ratio between terms

eg $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

$\frac{1/4}{1/2} = \frac{1}{2}$ $\quad \frac{1/8}{1/4} = \frac{1}{2}$ $\quad \frac{1/16}{1/8} = \frac{1}{2} \dots$

ratio r, first term a:

$$a, \; ar, \; ar^2, \; ar^3, \; \dots$$

$$\boxed{\sum_{k=0}^{N} r^k = r^0 + r^1 + r^2 + r^3 + r^4 + \dots + r^N = \frac{1-r^{N+1}}{1-r}}$$

Geometric sum

(ex) Simplify: $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}$

$$= \left(\tfrac{1}{3}\right)^0+\left(\tfrac{1}{3}\right)^1+\left(\tfrac{1}{3}\right)^2+\left(\tfrac{1}{3}\right)^3+\left(\tfrac{1}{3}\right)^4$$

$$= \sum_{k=0}^{4}\left(\tfrac{1}{3}\right)^k = \frac{1-(1/3)^5}{1-1/3}$$

$$= \frac{1-\frac{1}{3^5}}{\frac{2}{3}} = \frac{3}{2}\left(1-\frac{1}{3^5}\right) = \frac{3}{2}-\frac{1}{2\cdot 3^4}$$

(ex) Simplify $1+\frac{1}{9}+\frac{1}{9^2}+\frac{1}{9^3}+\frac{1}{9^4}+\cdots+\frac{1}{9^{100}}$

$$= \left(\tfrac{1}{9}\right)^0+\left(\tfrac{1}{9}\right)^1+\left(\tfrac{1}{9}\right)^2+\cdots+\left(\tfrac{1}{9}\right)^{100}$$

$$= \sum_{k=0}^{100}\left(\tfrac{1}{9}\right)^k = \frac{1-\left(\frac{1}{9}\right)^{101}}{1-\frac{1}{9}} = \left[1-\left(\tfrac{1}{9}\right)^{101}\right]\frac{9}{8}$$

ex Simplify $\frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81}$

$$= \frac{2}{3}\left[1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}\right]$$

$$= \frac{2}{3}\left[\left(\tfrac{1}{3}\right)^0 + \left(\tfrac{1}{3}\right)^1 + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^3\right]$$

$$= \frac{2}{3}\underbrace{\sum_{k=0}^{3}\left(\tfrac{1}{3}\right)^k}_{\text{formula}} = \frac{2}{3}\cdot\left[\frac{1-(1/3)^4}{1-1/3}\right]$$

$$\underbrace{\sum_{k=0}^{N} r^k}_{1+r+r^2+\cdots+r^N} = \frac{1-r^{N+1}}{1-r}$$

ex $\{a_n\}_{n=0}^{\infty} = \{r^n\}$

$$S_n = \sum_{k=0}^{n} a_k = \frac{1-r^{n+1}}{1-r}$$

Last time: $\lim_{n\to\infty} r^n \to 0$ if $|r|<1$; $\to$ DIV if $|r|>1$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1-r^{n+1}}{1-r}$$ const

$$\xrightarrow{\text{if } |r|<1} \frac{1-0}{1-r} = \frac{1}{1-r}$$

if $|r|>1$: DIV

ex Suppose $S_N = \frac{1}{2^N}$ ← size of $\underset{\text{(sum)}}{\underline{\text{pile}}}$ of N terms

Are the terms a_n positive or negative?

$S_1 = \frac{1}{2}$

$S_2 = \frac{1}{4}$

$S_3 = \frac{1}{8}$

$S_1 = a_1 = \frac{1}{2}$ (pos #)

$S_2 = a_1 + a_2$

$\frac{1}{4} = \frac{1}{2} + a_2 \rightarrow a_2 = \frac{-1}{4}$ (neg #)

$S_3 = \underbrace{a_1 + a_2}_{} + a_3$

$= S_2 + a_3$

$\frac{1}{8} = \frac{1}{4} + a_3 \qquad a_3 = \frac{-1}{8}$ (neg #)

a_1: pos

$a_n, n \geq 2$: neg

(ex) Suppose a_n is always positive, $(n \geq 1)$

and S_N is never greater than 1000.

Q1 : Is $\{S_N\}$ monotonic? a_1 $\quad a_1 + a_2$ $\quad a_1 + a_2 + a_3$ $\quad$ S_N always inc: $\boxed{\text{YES}}$

Q2 : Is $\{S_N\}$ bounded? $0 < S_N \leq 1000$ $\boxed{\text{YES}}$

Q3 : Is $\{S_N\}$ convergent YES

ie does $\lim_{N \to \infty} S_N$ exist, real # (not ∞)

Theorem: Every bounded, monotonic sequence converges.

Often: we can show a sum converges (ie limit exists) without knowing what it converges to (what is that limit?)

Ch. 8.3

A <u>series</u> (infinite sum) is defined much like an improper integral

$\{a_n\}$: sequence

series: $$\underbrace{\sum_{n=1}^{\infty} a_n}_{\text{How can you add } \infty \text{ many numbers?}} = \lim_{N\to\infty} \underbrace{\sum_{k=1}^{N} a_k}_{\text{finite sum — totally standard}} = \lim_{N\to\infty} S_N$$

Often it's fine to just imagine "adding up all the infinitely many terms"

Use definition to resolve ambiguous cases.

ex $\{a_n\}_1^\infty = \{(-1)^n\}_1^\infty$ sequence

$-1, 1, -1, 1, -1, 1, -1, 1$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \overset{\text{DEF}}{=} \lim_{N\to\infty} \sum_{k=1}^{N} (-1)^k = \lim_{N\to\infty} S_N \quad \underline{\text{DNE}}$$

The series <u>diverges</u>

$(-1+1) + (-1+1) + (-1+1) + \cdots = 0+0+0+\cdots = 0$

$-1 + (1-1) + (1-1) + (1-1) + \cdots = -1+0+0+0\cdots = -1$

$S_1 = -1$

$S_2 = 0$

$S_3 = -1$

$S_4 = 0$

Geometric Series

(computational interlude)

$$\sum_{n=0}^{\infty} ar^n = \lim_{N\to\infty} \underbrace{a\sum_{n=0}^{N} r^n}_{\text{partial sum}}$$

$$= \lim_{N\to\infty} a\frac{1-r^{N+1}}{1-r} \xrightarrow{|r|<1} a\frac{1}{1-r}$$

$$\xrightarrow{|r|>1} \text{DIVERGES}$$

ex

$$\sum_{n=0}^{\infty} (1/5)^n = 1+\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\cdots$$

$r = 1/5$

$a = 1$

$$= \frac{1}{1-1/5}$$

ex $\displaystyle\sum_{k=0}^{\infty} \left(-\frac{10}{9}\right)^n$ DIV

Note: $r = -\frac{10}{9}$

$|r| = \frac{10}{9} > 1$

ex $\displaystyle\sum_{k=2}^{\infty} \left(\frac{2}{3}\right)^n$

$r = \frac{2}{3}$, $|r| < 1$

$$= \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^4 + \cdots$$

$$= \left(\frac{2}{3}\right)^2 \left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \cdots\right] = \left(\frac{2}{3}\right)^2 \underbrace{\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n}$$

$$= \boxed{\frac{4}{9} \cdot \frac{1}{1 - 2/3}}$$

ex $\displaystyle\sum_{n=2}^{\infty} \frac{10^n}{9^{2n}} = \sum_{n=2} \frac{10^n}{81^n} = \sum_{n=2}^{\infty} \left(\frac{10}{81}\right)^n$

$r = \frac{10}{81}$ will conv

$$= \left(\frac{10}{81}\right)^2 + \left(\frac{10}{81}\right)^3 + \left(\frac{10}{81}\right)^4 + \cdots$$

- could factor out first term
- could also add/subtract

$$= \underbrace{1 + \frac{16}{81} + \left(\frac{16}{81}\right)^2 + \left(\frac{16}{81}\right)^3 + \left(\frac{16}{81}\right)^4 + \cdots}_{\text{what I was given}} \quad \text{can evaluate}$$

$$-1 - \frac{16}{81}$$

$$= \underbrace{\sum_{n=0}^{\infty} \left(\frac{10}{81}\right)^n}\; -1 - \frac{16}{81}$$

$$= \boxed{\frac{1}{1 - 10/81} - 1 - \frac{16}{81}}$$

$3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 =$
$3 \cdot (3^4)$

$x^{a+b} =$
$x^a x^b$

ex

$$\sum_{n=5}^{\infty} \frac{3^{n+1}}{2^{2n}} = \sum_{n=5}^{\infty} \frac{3 \cdot 3^n}{4^n} = 3 \sum_{n=5}^{\infty} \left(\frac{3}{4}\right)^n$$

$r = 3/4$
$|r| < 1$
so will
conv

$$3 \sum_{n=5}^{\infty} \left(\frac{3}{4}\right)^5 \left(\frac{3}{4}\right)^{n-5} = 3 \cdot \left(\frac{3}{4}\right)^5 \sum_{n=5}^{\infty} \left(\frac{3}{4}\right)^{n-5}$$

$$= 3\left(\frac{3}{4}\right)^5 \underbrace{\sum_{k=0}^{\infty}} \left(\frac{3}{4}\right)^k = \boxed{3\left(\frac{3}{4}\right)^5 \frac{1}{1 - 3/4}}$$

Set $k = n-5$
when $n=5$,
$k = 5-5 = 0$

Ch 8.4 Divergence & Integral Tests

DIV OR CONV?

$$\sum_{n=1}^{\infty} 1 = \lim_{K\to\infty} \underbrace{\sum_{n=1}^{K} (1)} = \infty \quad \text{DIV}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{10} + \frac{1}{2n} \right) = \infty \quad \text{DIV}$$

$$\frac{1}{10} + (\ ) + \frac{1}{10} + (\ ) + \frac{1}{10} + (\ ) + \frac{1}{10} + (\ ) + \cdots$$

$$\sum_{n=1}^{\infty} (-1)^n \quad \text{DIV}$$

DIVERGENCE TEST:

If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=a}^{\infty} a_n$ DIVERGES

Sequence $\{a_n\}$

$\lim_{n\to\infty} a_n \neq 0$

For sure, $\sum_{n=a}^{\infty} a_n$ diverges

$\lim_{n\to\infty} a_n = 0$

Maybe $\sum_{n=a}^{\infty} a_n$ DIV, maybe CONV

we'll learn <u>tests</u> to decide