Recall from Last Time:

A <u>sequence</u> is a list of numbers

eg $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ is the list

$\frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots$

A <u>series</u> is the sum of terms in a sequence

eg $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$

We saw that this sum gets closer and closer to $\boxed{1}$ as we add more terms, while the terms themselves approach $\boxed{0}$.

Vocabulary

A sequence is **bounded** if there exist real numbers a and b such that terms of sequence all lie between a & b

a – "floor"   b – "ceiling"

all terms lie btw them
never bigger than b
never smaller than a

A sequence is **monotonic** if it's never decreasing or if it's never increasing

eg. 1, 1, 2, 2, 3, 3, 4, 4   never dec, so monotonic

eg. 1 2 1 2 1 2 1 2   not monotonic

A sequence $\{a_n\}$ converges if $\lim_{n\to\infty} a_n = L$ for some real number L,
diverges otherwise.

$\boxed{\text{Theorem}}$ Every bounded, monotonic sequence converges.

e.g. Box of money, you never take any out, it never has more than \$1000

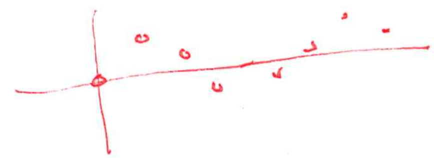
a_n : amt \$ in box at time n

$\{a_n\}$ bounded : $0 \le a_n \le 1000$ for all n

$\{a_n\}$ monotonic : never take money out, ie $\{a_n\}$ never decreases

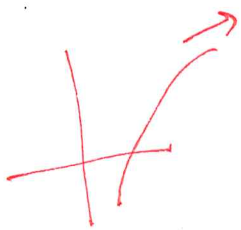
So $\{a_n\}$ converges

function	bdd?	mono?	conv?
$\{a_n\} = \{\sin n\}$	Y	N	N
$\{b_n\} = \{\ln n\}$	N	Y	N
$\{c_n\} = \{e^{-n}\} = \{\frac{1}{e^n}\}$	Y	Y	Y
$\{d_n\} = \{\frac{\sin n}{n}\}$	Y	N	Y



a_n: $-1 \leq \sin n \leq 1$

b_n:

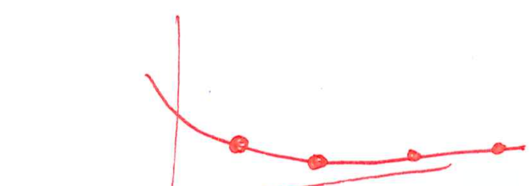


increasing

$\lim_{n\to\infty} \ln n = \infty$

n starts at some finite value n_0

c_n:

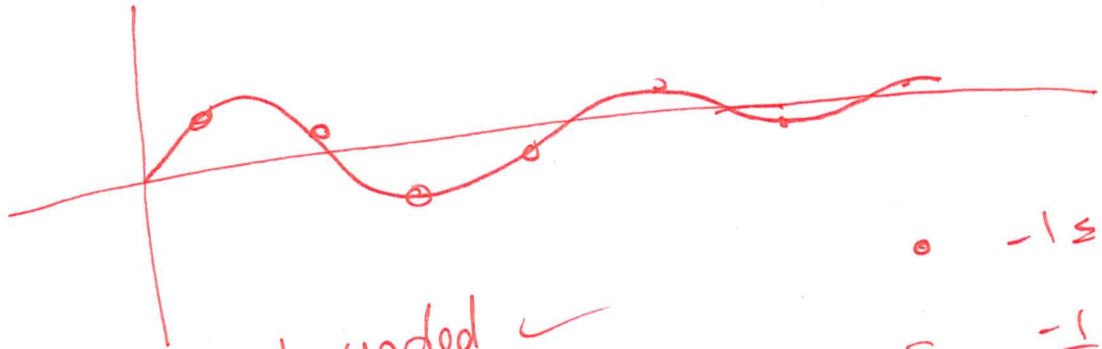


decreasing

$\lim_{n\to\infty} e^{-n} = 0$

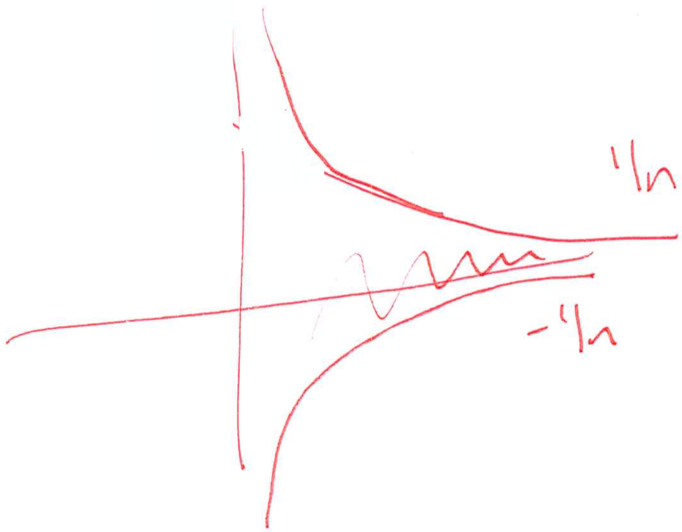
$0 \leq \frac{1}{e^n} \leq \frac{1}{e^{n_0}}$

a_n:



bounded ✓

monotonic ✗



• $-1 \le \sin n \le 1$

So $\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

• $\lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$

So by squeeze theorem,

$\lim_{n\to\infty} \frac{\sin n}{n} = 0$

A geometric sequence has a common ratio between its terms

e.g. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$ $\left\{\left(\frac{1}{2}\right)^n\right\}$

$\frac{1/4}{1/2} = \frac{2}{4} = \frac{1}{2}$

$\frac{1}{8} / \frac{1}{4} = \frac{4}{8} = \frac{1}{2}$

$\frac{1/16}{1/8} = \frac{8}{16} = \frac{1}{2}$

$\frac{1/32}{1/16} = \frac{16}{32} = 1/2$

Recursively:

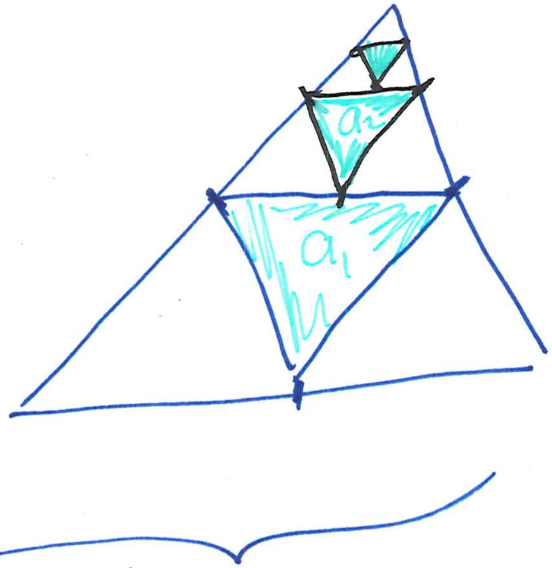
$a_0 = a$ ratio: r

$a_n = r a_{n-1}$

Explicit:

$a_n = a r^n$

ex



1

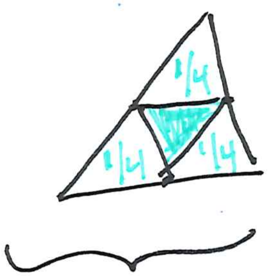
$a_1 = \frac{1}{4}$

$a_2 = \frac{1}{16}$

$a_3 = \frac{1}{64}$

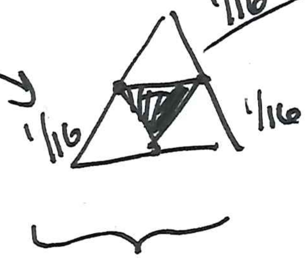
$r = 1/4$

$a_n = 1 \cdot \left(\frac{1}{4}\right)^n = \frac{1}{4^n}$



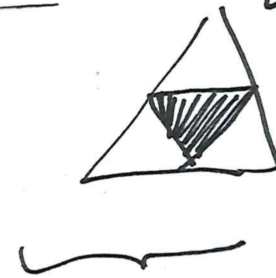
1

$a_1 = \frac{1}{4}$



1/4

$a_2 = \frac{1}{16}$



1/16

$\frac{1}{4}\left(\frac{1}{16}\right) = \frac{1}{64}$

(ex) Computer storage (avg machine) increases 40% per year

$t=0$: avg storage 1 GB

$\{a_t\}$: storage year t

$$a_t = a_{t-1} + 0.4a_{t-1} = 1.4\,a_{t-1}$$

$\boxed{r=1.4}$

So: $a_t = (1.4)^t$

$a_0 = 1.4^0 = 1$ GB

$a_1 = 1.4^1 = 1.4$ GB

$a_2 = 1.4^2$ GB

$\vdots$

Partial Sums (sequence)

Given a sequence $\{a_n\}$, its sequence of partial sums $\{S_n\}$ is:

$$S_n = \sum_{k=1}^{n} a_k$$

S_n: sum of all terms up to n^{th}

e.g. $\{a_n\} = \{1\}$

a_n: 1, 1, 1, 1, 1, 1, ...

S_n: 1, 2, 3, 4, ... partial sums

a_1 a_1+a_2 $a_1+a_2+a_3$ $a_1+a_2+a_3+a_4$

(ex) $\{a_n\} = \left\{ \frac{1}{2^n} \right\}$

$\{s_n\}$ partial sums: $S_N = \sum_{n=1}^{N} a_n$

$a_1 = \frac{1}{2}$ $\quad S_1 = \frac{1}{2}$

$a_2 = \frac{1}{4}$ $\quad S_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

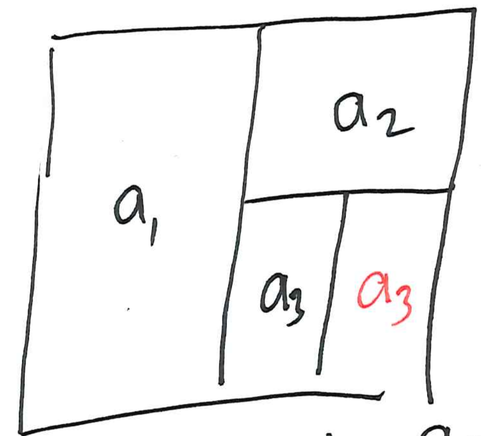
$a_3 = \frac{1}{8}$ $\quad S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 1 - \frac{1}{8} = \frac{7}{8}$

$a_4 = \frac{1}{16}$ $\quad S_4 = 15/16 = 1 - 1/16$

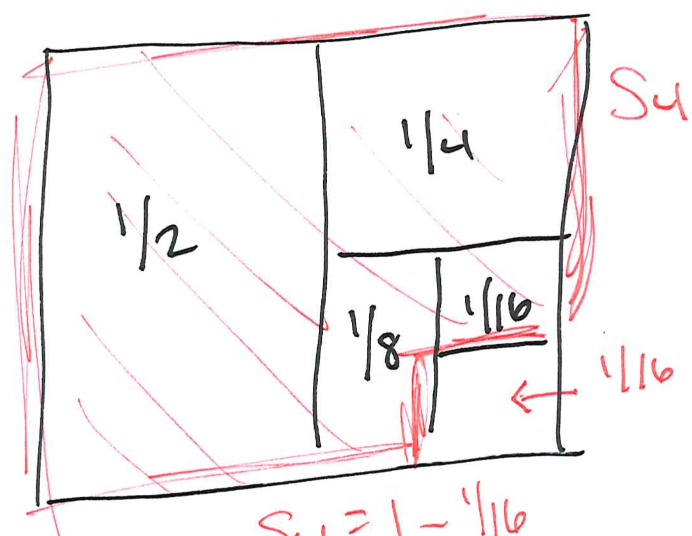
$a_5 = \frac{1}{32}$ $\quad S_5 = 1 - \frac{1}{2^5} = \frac{31}{32}$

$\vdots$

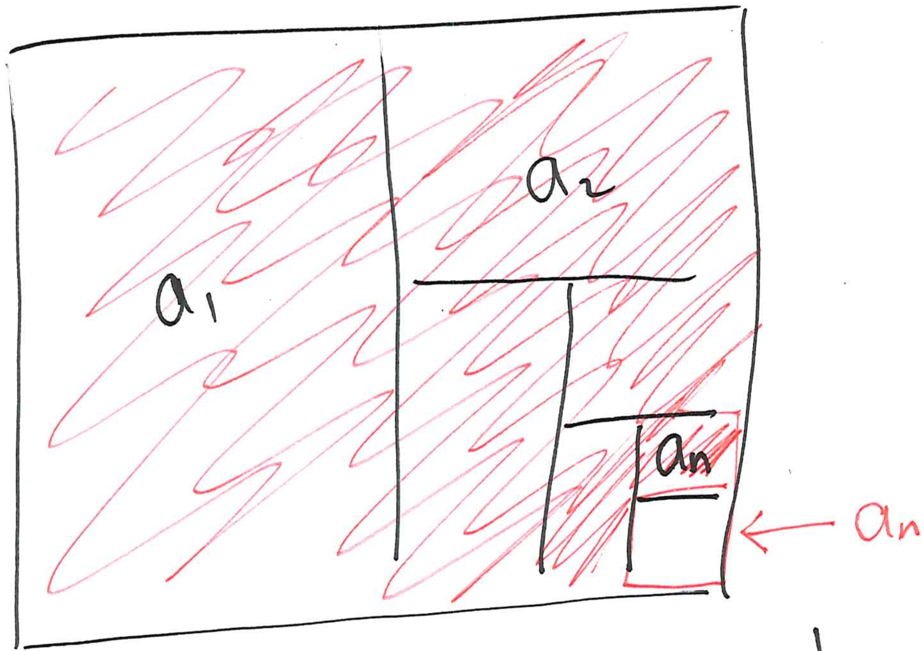
$a_n = \frac{1}{2^n}$ $\quad S_n = 1 - \frac{1}{2^n}$



$a_1 + a_2 + a_3 = 1 - a_3$
$= 1 - \frac{1}{2^3}$



$S_4 = 1 - 1/16$



$S_n = a_1 + a_2 + \cdots + a_n$

$= 1 - a_n =$

$\boxed{1 - \frac{1}{2^n}}$

Sequence: size of tiles $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

$$\lim_{n \to \infty} a_n = 0$$

Sequence of partial sums: $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots$

$$\lim_{n \to \infty} S_n = 1$$

Still same series

evaluate $S_4 - S_3 = (a_1 + a_2 + a_3 + a_4) - (a_1 + a_2 + a_3) = a_4 = \frac{1}{2^4}$

evalute $S_{15} - S_{14} = a_{15} = \frac{1}{2^{15}}$

add up first 15 terms

subtract off first 14 terms

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ex) Suppose $S_N = \sum_{k=1}^{N} \frac{1}{10^k}$

$S_1 = \frac{1}{10} = 0.1$

$S_2 = \frac{1}{10} + \frac{1}{10^2} = 0.11$

$S_3 = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} = 0.111$

$S_4 = 0.1111$

$S_5 = 0.11111$

$\lim_{n \to \infty} S_n = 0.\overline{1} = \boxed{1/9}$

"S_∞": $0.1111111111111111111111111111111$

$0.\overline{1}$

(ex) Suppose $S_N = \sum_{k=1}^{N} a_k = N^2+1$

- What is a_{100}?
- What is a_1?
- What is a_n?

$a_{100} = 199$

$a_1 = 2$

$S_1 = \sum_{k=1}^{1} a_k = a_1 = 1^2+1 = 2$

$S_2 = \sum_{k=1}^{2} a_k = a_1 + a_2 = 2^2+1$

$$\begin{array}{r} a_1 + a_2 = 5 \\ a_1 = 2 \\ \hline a_2 = 3 \end{array}$$

$$\begin{array}{l} S_{100} - S_{99} = (a_1 + a_2 + \cdots + a_{99} + a_{100}) \\ \qquad - (a_1 + a_2 + \cdots + a_{99}) \\ \hline \end{array}$$

$S_{100} \to 100^2+1$

$S_{99} \to 99^2+1$

$= a_{100}$

$= (100^2+1) - (99^2+1)$

$= 100^2 - 99^2$

$= (100+99)\cancel{(100-99)}^{1}$

$= 199$

$$a_n = S_n - S_{n-1} = (n^2+1) - ((n-1)^2+1)$$

$$= n^2 - (n-1)^2 = (n+(n-1))\overset{1}{\cancel{(n-(n-1))}}$$

$$= 2n-1$$

$a_1 = 2$

$a_2 = 3$

$a_3 = 5$

$a_4 = 7$

$\vdots$