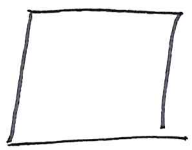
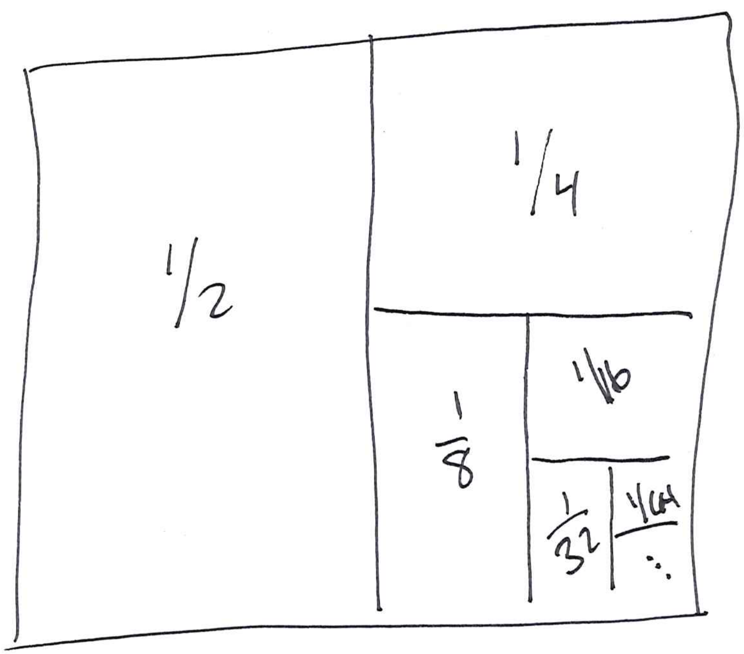
New Topic: Sequences & Series

Sequences: $\underbrace{\text{list of numbers}}_{\text{LIST}}$

Series: $\underbrace{\text{add up numbers in a sequence}}_{\text{SUM (\#)}}$

ex  A: 1 area of square is 1



List of Areas:

$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \dots$

(sequence) Notice: terms $\to 0$

Sum all those areas:

$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = 1$

(series)

Ch 8.1

Sequence: list of numbers

- function whose domain is whole numbers

e.g. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

1ST, 2nd, 3rd, 4th ...

$f(1)=\frac{1}{2}$, $f(2)=\frac{1}{4}$, $f(3)=\frac{1}{8}$, $f(4)=\frac{1}{16}$ ...

BUT: we don't define $f(0.1)$, $f(1/2)$, etc.

- Notation: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

$a_1, a_2, a_3, a_4, \ldots$

$\{a_n\}_{n=1}^{\infty}$ means sequence $a_1, a_2, a_3, \ldots$

can give a rule e.g. $\{a_n\}_{n=1}^{\infty} = \{\frac{1}{2^n}\}_{n=1}^{\infty}$

100th term: $a_{100} = \frac{1}{2^{100}}$
$(n = 100)$

If we don't care much about first terms we omit bounds

eg $\{a_n\} = \{\frac{1}{2^n}\}$

we understand: a_n starts at some n, goes to ∞ many terms

Two main ways to describe sequences:

recursive

Give starting pt
Need prev term to find next term

explicit

$a_n = f(n)$
You only need to know n to find a_n

Fibonacci:

$F_0 = 1$

$F_1 = 1$

If $n \geq 2$:

$F_n = F_{n-1} + F_{n-2}$

before

two before

Recursive

To find F_5, need to know F_1, F_2, F_3, F_4!

$F_0 = 1$

$F_1 = 1$

$F_2 = 2$

$F_3 = 3$

$F_4 = 5$

$\boxed{F_5 = 8}$

ex $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots$

Explicit: $\{a_n\}_{n=1}^{\infty} = \left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$

To find a_{10}, don't need to find $a_1, a_2, \dots, a_9$

$$\boxed{a_{10} = \frac{1}{2^{10}}}$$

Recursive: $a_1 = \frac{1}{2}$

$a_n = \frac{1}{2} a_{n-1}$ if $n \geq 2$

To find a_5:

$a_1 = \frac{1}{2}$

$a_2 = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

$a_3 = \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8}$

$a_4 = \frac{1}{2}\left(\frac{1}{8}\right) = \frac{1}{16}$

$a_5 = \frac{1}{2}\left(\frac{1}{16}\right) = \boxed{\frac{1}{32}}$

ex $\frac{-1}{3}, \frac{1}{9}, \frac{-1}{27}, \frac{1}{81}, \ldots$

explicit: $\{a_n\} = \left\{\left(\frac{-1}{3}\right)^n\right\}$

recursive: $a_1 = \frac{-1}{3}$

$a_n = \frac{-1}{3} a_{n-1}$

ex $\frac{1}{4}, \frac{1}{10}, \frac{1}{28}, \frac{1}{82}, \ldots$

explicit:

$\{a_n\} = \left\{\frac{1}{3^n+1}\right\}$

Ch 8.2: Sequences

Limit Laws

$\boxed{\text{Thm}}$ Suppose f is a function such that $f(n) = a_n$ for all whole numbers n. If

$$\lim_{x\to\infty} f(x) = L,$$

then also

$$\lim_{n\to\infty} a_n = L.$$

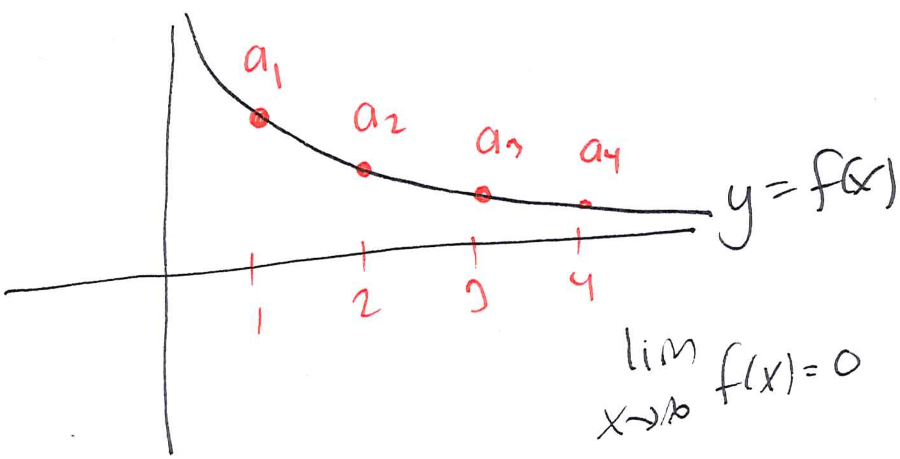
—

(ex) $\{a_n\} = \left\{ \frac{1}{3^n+1} \right\}$

$f(x) = \frac{1}{3^x+1}$ defined for all reals

If n whole number:

$a_n = f(n)$



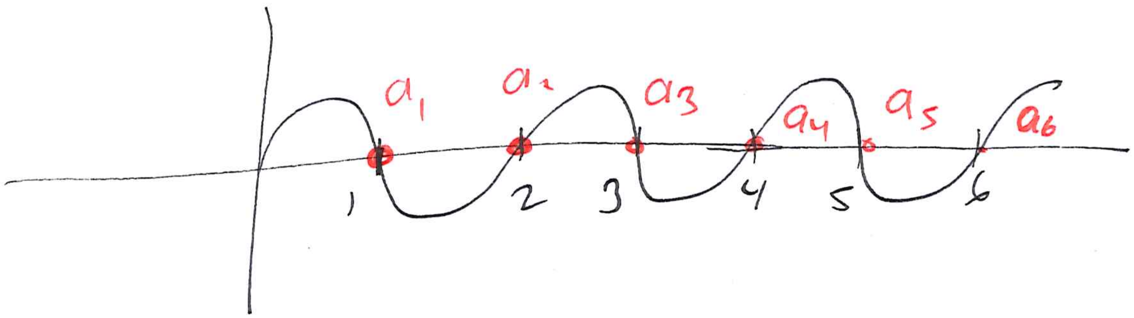
$\lim_{x\to\infty} f(x) = 0$

So: $\lim_{n\to\infty} a_n = 0$

(ex) If $f(n) = a_n$, but $\lim_{x\to\infty} f(x)$ DNE
thm above does not apply.
Maybe $\lim_{n\to\infty} a_n$ DNE, maybe $\lim_{n\to\infty} a_n$ exists.

Let $f(x) = \sin(\pi x)$

Let $a_n = f(n)$



Sequence:
$0, 0, 0, 0, 0, 0, \ldots$

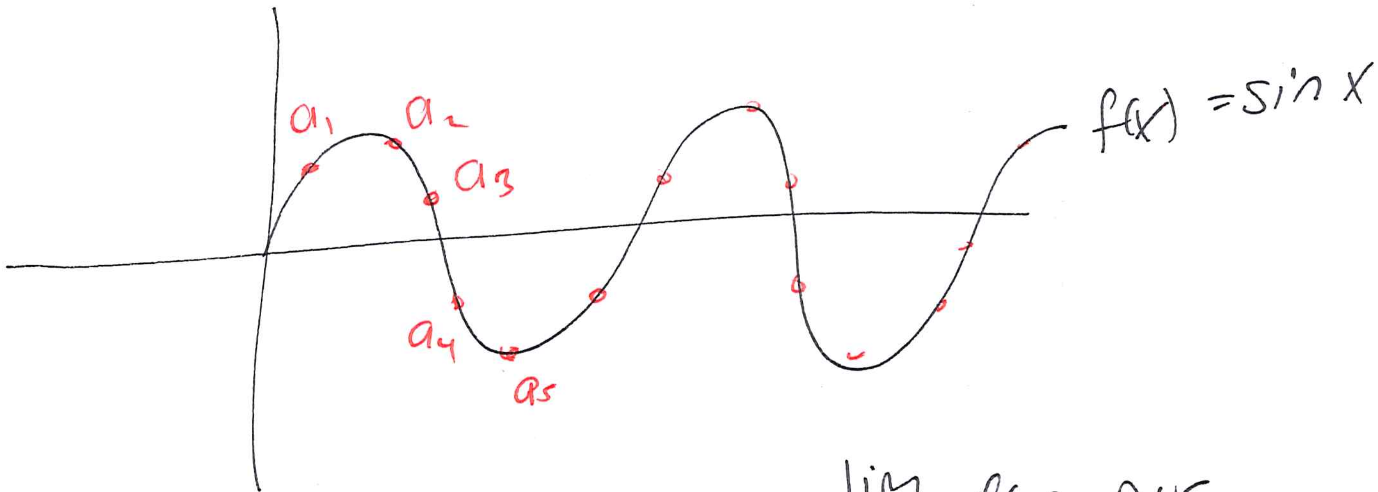
$\lim_{x\to\infty} f(x)$ DNE

$\lim_{n\to\infty} a_n = 0$

We're only guaranteed $\lim_{x\to\infty} f(x) = \lim_{n\to\infty} a_n$ if
$\lim_{x\to\infty} f(x)$ exists.

otherwise MAYBE sequence cherry-picks parts of the function.

ex $f(x) = \sin x$ $\{a_n\} = \{\sin n\}$



In this case: $\lim_{x \to \infty} f(x)$ DNE

AND $\lim_{n \to \infty} a_n$ DNE

Limit Laws for Sequences

Suppose $\{a_n\}$, $\{b_n\}$ sequences, and

$$\lim_{n\to\infty} a_n = A, \quad \lim_{n\to\infty} b_n = B$$

1. $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$
2. $\lim_{n\to\infty} (c\, a_n) = cA$, c constant
3. $\lim_{n\to\infty} (a_n b_n) = AB$
4. $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = A/B$ if $B \neq 0$

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ex $\lim_{n\to\infty} \left(\frac{1}{2n}\right) = 0$, $\lim_{n\to\infty} \left(\frac{n}{n+1}\right) = 1$

So: $\lim_{n\to\infty} \left(\frac{1}{2n} + \frac{n}{n+1}\right) = 0 + 1 = 1$

If limits DNE, need to be more careful

ex

$\{a_n\} = \frac{1}{n^2}$

$\frac{1}{1}, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$

$\lim_{n\to\infty} a_n = 0$

$\{b_n\} = 2n^2$

$2, 8, 18, 34, \ldots$

$\lim_{n\to\infty} b_n = \infty$ (DNE)

$\{c_n\} = \{a_n b_n\} = \{\frac{1}{n^2} \cdot 2n^2\} = \{2\}$

$1 \cdot 2, \frac{1}{4} \cdot 8, \frac{1}{9} \cdot 18, \frac{1}{16} \cdot 34, \ldots$

$2, 2, 2, 2, \ldots$

$\lim_{n\to\infty} (a_n b_n) = 2$

Squeeze Thm for Sequences

(often used when we can't use limit laws because limit of a piece DNE)

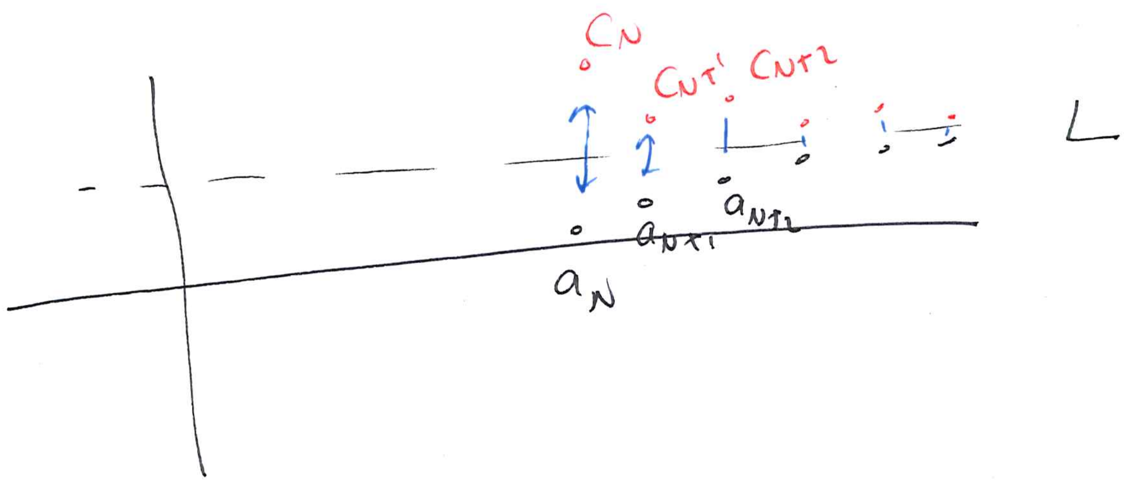
Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences with

$$a_n \le b_n \le c_n$$

for all $n \ge N$ for some fixed N.

If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then also

$$\boxed{\lim_{n\to\infty} b_n = L}$$



I know limits of $\{a_n\}$, $\{c_n\}$

Thm tells me about lim of $\{b_n\}$

ex. Evaluate $\lim_{n\to\infty} b_n$ where $\{b_n\} = \left\{\frac{2n+\cos n}{n+1}\right\}$

Goal: deal with cosine using squeeze theorem.

Note: $-1 \le \cos n \le 1$

So: $\frac{2n-1}{n+1} \le \frac{2n+\cos n}{n+1} \le \frac{2n+1}{n+1}$

Need to show: $\lim_{n\to\infty} \frac{2n-1}{n+1} = \lim_{n\to\infty} \frac{2n+1}{n+1}$

Rational fcn: deg of top = deg of bottom

So $\lim_{n\to\infty}$ ratio of leading coeff

$\lim_{n\to\infty} \frac{2n-1}{n+1} = \frac{2}{1} = 2$, $\lim_{n\to\infty} \frac{2n+1}{n+1} = \frac{2}{1} = 2$

By Squeeze Thm, also

$$\boxed{\lim_{n\to\infty} b_n = 2}$$

Theorem: Any bounded, monotonic sequence converges.

convergent: $\lim_{n\to\infty}$ exists (not $\pm\infty$)

bounded: there are constants "a" and "c"
(like floor + ceiling)
all terms of our sequence are between a + c

eg No term of $\{a_n\}$ is bigger than 1,000
and no term is smaller than -10

monotonic: never decreasing OR never increasing

We use this thm in series

(ex) You never take money out of your account,
it never gets more than \$1000.

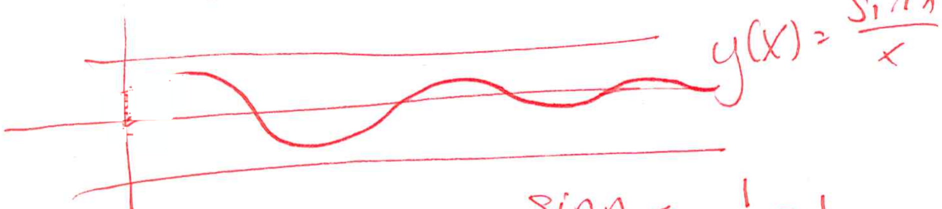
$\{A_n\}$: account balance at time n

Bounded: $A_n \le 1000$
and $A_n \ge A_0$

Monotonic: A_n never decreases

So by thm: $\lim_{n\to\infty} A_n = L$, some real number L

Sequence	bdd?	mon?	conv?
$a_n = \sin n$	YES $-1 \le \sin n \le 1$	NO $\nearrow\searrow$	NO
$b_n = \ln n$	NO	YES $\nearrow$	NO $\lim_{n\to\infty} \ln n = \infty$
$c_n = e^{-n} = \frac{1}{e^n}$ $n \ge 0$	YES $e^0 \ge e^{-n} > 0$	YES $\searrow$	YES
$d_n = \cancel{\sin n}\ \frac{\sin n}{n}$ $(n \ge 1)$	YES $-1 \le \sin n \le 1$ $-1 \le n$	NO	YES $\lim_{n\to\infty} \frac{\sin n}{n} = 0$



So $\frac{\sin n}{n} \le \frac{1}{1} = 1$

$\frac{-1}{1} \le \frac{\sin n}{n}$

Can prove using Squeeze Thm

$\frac{-1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$

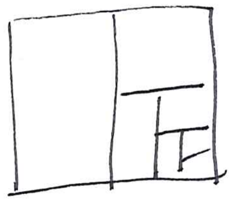
$\lim_{n\to\infty} \frac{-1}{n} = 0 = \lim_{n\to\infty} \frac{1}{n}$

So $\lim_{n\to\infty} \frac{\sin n}{n} = 0$ also

Geometric Sequence

Ratio btw consecutive terms constant

e.g.



$\frac{1}{2} \quad \frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{16} \quad \frac{1}{32} \quad \ldots$

$\frac{1}{2} / \frac{1}{4} = 2$ $\quad$ $\frac{1}{4} / \frac{1}{8} = 2$ $\quad$ $\frac{1}{8} / \frac{1}{16} = 2$ $\quad$ $\frac{1}{16} / \frac{1}{32} = 2 \ \ldots$

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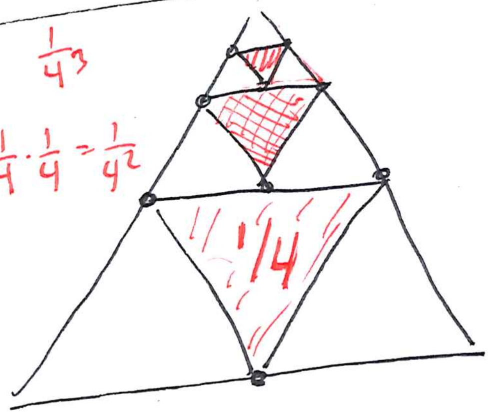
eg

$\frac{1}{4^3}$

$\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^2}$



$A = 1$

Sequence: $\frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{4^4}, \ldots$

Common Ratio: $\frac{1}{4}$

eg Storage capacity of avg computer increases 40% every year.

Suppose in 2018, avg computer has 500 GB storage.

$\{a_n\}_{n=2018}^{\infty} = \{\text{avg storage year } n\}_{n=2018}^{\infty}$

$a_{2018} = 500$

Rule:

$$\left.\begin{aligned} a_n &= a_{n-1} + 0.4\, a_{n-1} \\ &= 1.4\, a_{n-1} \end{aligned}\right) \text{recursive}$$

$a_n = 500$ if $n = 2018$

$a_n = 1.4^{n-2018} \cdot 500$ if $n > 2018$ explicit

Geometric Sequence, common ratio: 1.4

$a_{2018} = 500$

$a_{2019} = 1.4 \cdot 500$

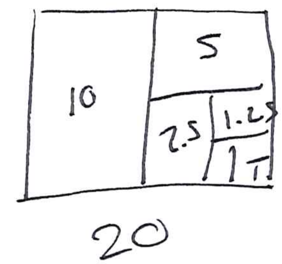
$a_{2020} = 1.4 \cdot 1.4 \cdot 500$

$\vdots$

limits of geom sequences → $a_n = a \cdot r^n$
some fixed r

$$\lim_{n\to\infty} 20\left(\tfrac{1}{2}\right)^n = 0$$

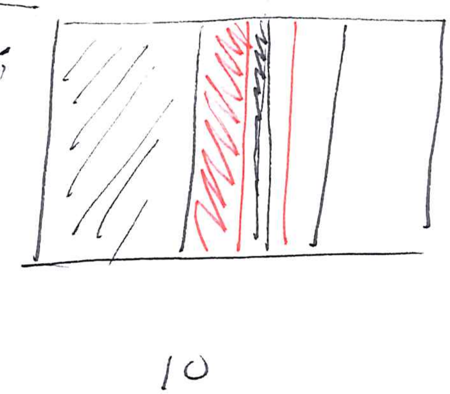
$r = 1/2$



$$\lim_{n\to\infty} 10\left(\tfrac{-1}{3}\right)^n = 0$$

$r = -1/3$

$$\lim_{n\to\infty} 4(-1.1)^n \text{ DNE}$$



$$\lim_{n\to\infty} (1.001)^n = \infty$$

$$\lim_{n\to\infty} ar^n = \begin{cases} 0 & \text{if } |r| < 1 \ \circledast \\ a & \text{if } r = 1 \\ \text{DNE unless } a = 0 & \text{if } r = -1 \\ \text{DIV} & \text{if } |r| > 1 \ \circledast \end{cases}$$

$r = 1$ → $\lim_{n\to\infty} a \cdot 1^n = \lim_{n\to\infty} a$

$r = -1$ → $\lim_{n\to\infty} a(-1)^n$

$a, -a, a, -a, a, -a, \ldots$