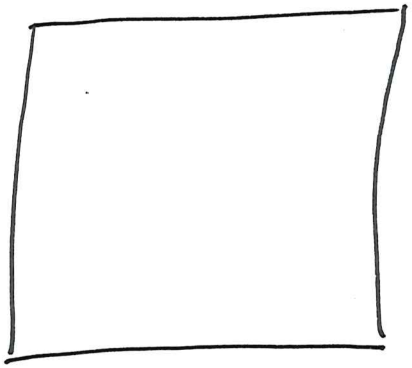
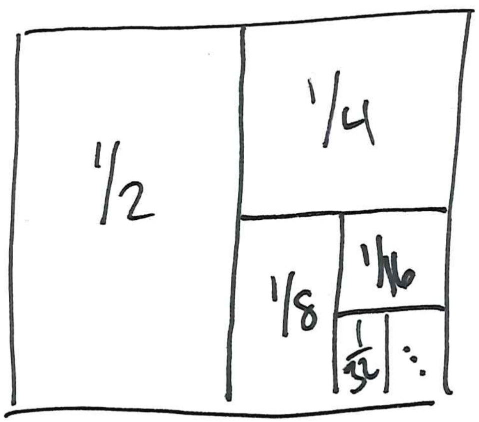
Sequences & Series

lists of numbers

sums of those lists



Area: 1



Segment areas:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$$

$$\lim_{n \to \infty} A_n = 0$$

sequence

Adding:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = 1$$

series

Sometimes, we don't want to add sequence terms — leave as a list

Ch 8.1

Sequence: function whose domain is whole #s

ex:

3 689 257	$= f(1)$	1st number	NC $f(1/2)$
4 324 810	$= f(2)$	2nd number	
4 833 239	$= f(3)$	3rd number	
$\vdots$	$\vdots$		

Notation: a_n : n^{th} term

explicit rule: $\{a_n\}_{n=1}^{\infty} = \{f(n)\}_{n=1}^{\infty}$

ex: If I write $\{a_n\}_{n=0}^{\infty} = \{100+5n\}_{n=0}^{\infty}$

that gives sequence

$a_0 = 100$

$a_1 : 100+5 = 105$

$a_2 : 110$

$\vdots$

Often, if we only care about long-term behaviour, we omit bounds, understand sequence starts somewhere, continues infinitely long

ex: $\{a_n\} = \{\frac{1}{2^n}\}$

Two ways to describe sequences:

<u>recursive</u>

Find a term using previous terms

<u>explicit</u>

Find a term by knowing which it is (eg 4^{th} term, 100^{th} term, etc)

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ex Bank account, starts with \$100
deposit \$5 each month
(no interest, no withdrawals)

Sequence

$100 = f(0)$
$105 = f(1)$
$100 = f(2)$
115
120
$\vdots$

<u>Recursive</u> $f(0) = 100$

$$f(n) = f(n-1) + 5$$

<u>Explicit</u> $f(n) = 100 + 5n$

ex Write explicit formula:

1. $\overset{a_1}{\frac{1}{2}}, \overset{a_2}{\frac{1}{4}}, \overset{a_3}{\frac{1}{8}}, \overset{a_4}{\frac{1}{16}}, \ldots$

$$\{a_n\} = \left\{\frac{1}{2^n}\right\}$$

2. $\overset{b_1}{\frac{-1}{3}}, \overset{b_2}{\frac{1}{9}}, \overset{b_3}{\frac{-1}{27}}, \overset{b_4}{\frac{1}{81}}, \ldots$

$$\{b_n\} = \left\{\left(\frac{-1}{3}\right)^n\right\}$$

3. $\frac{1}{3}, \frac{1}{5}, \frac{1}{9}, \frac{1}{17}, \ldots$

$$\{c_n\} = \left\{\frac{1}{2^n+1}\right\}$$

Recursive:

$a_1 = \frac{1}{2}$

$a_n = \frac{1}{2} a_{n-1}$

$b_1 = \frac{-1}{3}$

$b_n = \frac{-1}{3} b_{n-1}$

hard to get recursive form

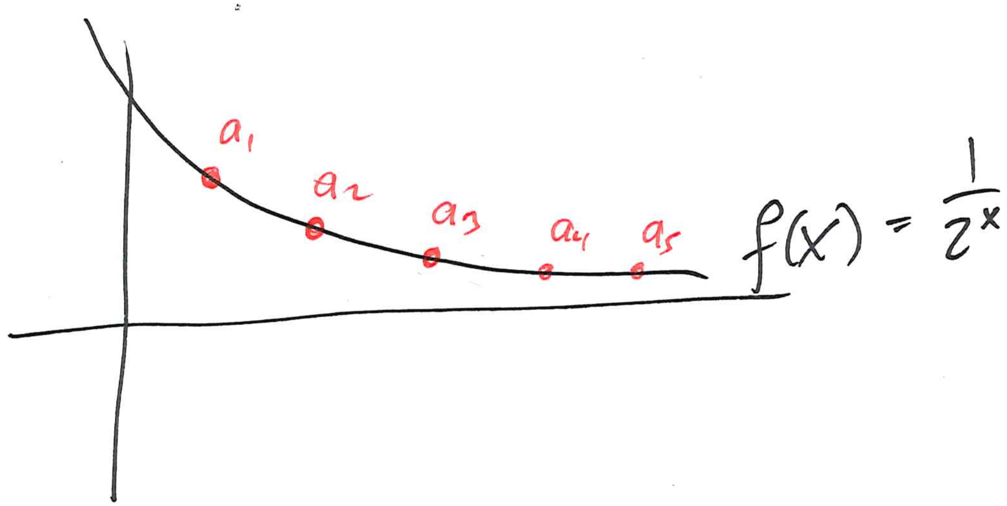
ex $\{a_n\} = \{\frac{1}{2^n}\}$: evaluate $\lim_{n\to\infty} a_n$

$$= \lim_{n\to\infty} \frac{1}{2^n} = 0$$

Sometimes, not quite always, limit of sequence looks like limit of function

$\boxed{\text{Limit Laws}}$

Suppose f is a function with $f(n) = a_n$ for all positive integers n. If $\lim_{n\to\infty} f(n)$ exists and is L, then $\lim_{n\to\infty} a_n = L$.



ex

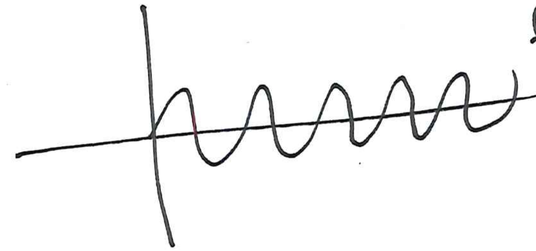
$$\{b_n\} = \{\sin(\pi n)\}$$

$$\lim_{n\to\infty} b_n \ ?$$

Note:

$\lim_{n\to\infty} \sin(\pi n)$ DNE

$f(x) = \sin(\pi x)$



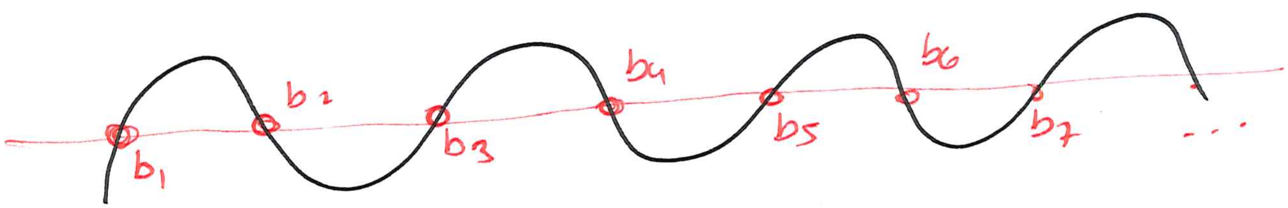
$$b_1 = \sin(\pi) = 0$$
$$b_2 = \sin(2\pi) = 0$$
$$b_3 = \sin(3\pi) = 0$$
$$b_4 = \sin(4\pi) = 0$$
$$\vdots$$

Sequence: $0, 0, 0, 0, 0, 0, 0, \ldots$

So $\boxed{\lim_{n\to\infty} b_n = 0}$

We cherry-picked our values

$f(x) = \sin(\pi x)$



$\{a_n\} = \{f(n)\}$

only exists if n whole number

exists for all real n

If $\lim_{n\to\infty} f(n)$ DNE,

$\lim_{n\to\infty} a_n$ unclear

May exist, may not

Need to look further

If $\lim_{n\to\infty} f(n) = L$

then

$\lim_{n\to\infty} a_n = L$

More Limit Laws: $\{a_n\}$, $\{b_n\}$ sequences with

$\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$

① $\lim_{n\to\infty} (a_n \pm b_n) = A \pm B$

② $\lim_{n\to\infty} (c a_n) = cA$, c: constant

③ $\lim_{n\to\infty} (a_n b_n) = AB$

④ $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ if $B \neq 0$

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ex) $\{a_n\} = \left\{\frac{n^2+3}{n^2+1}\right\}$, $\{b_n\} = \{2\arctan(n)\}$

Consider $\{a_n b_n\}$

New Sequence:

$a_1 = \frac{4}{2}$ $b_1 = 2\arctan(1)$ $c_1 = \frac{4}{2}\cdot 2\arctan 1$

$a_2 = \frac{7}{5}$ $b_2 = 2\arctan(2)$ $c_2 = \frac{7}{5}\cdot 2\arctan(2)$

$a_3 = \frac{12}{10}$ $b_3 = 2\arctan(3)$ $c_3 = \frac{12}{10}\cdot 2\arctan(3)$

$\vdots$

$\lim_{n\to\infty} a_n = 1$ $\lim_{n\to\infty} b_n = 2(\pi/2) = \pi$

Limit Law:

$\lim_{n\to\infty} (a_n b_n) = (1)(\pi) = \pi$

Recall: a rational function $\frac{p(x)}{q(x)}$

$\deg(p) = n$

$\deg(q) = m$

If $n=m$: $\lim_{x \to \infty} \frac{p(x)}{q(x)} =$ ratio of leading coeff

If $n<m$: $\lim_{x \to \infty} \frac{p(x)}{q(x)} = 0$

If $n>m$: $\lim_{x \to \infty} \frac{p(x)}{q(x)} = \pm\infty$

ex: $\lim_{x \to \infty} \frac{x^2 + 2x + 5}{3x^2 + 9} = \frac{1}{3}$

ex)

$\{a_n\} = \left\{\frac{1}{n^2}\right\}$, $\{b_n\} = \{2n^2\}$

$\lim_{n\to\infty} a_n = 0$

$\underbrace{\lim_{b\to\infty} b_n = \infty}_{\text{DNE}}$

CANNOT SIMPLY SAY:

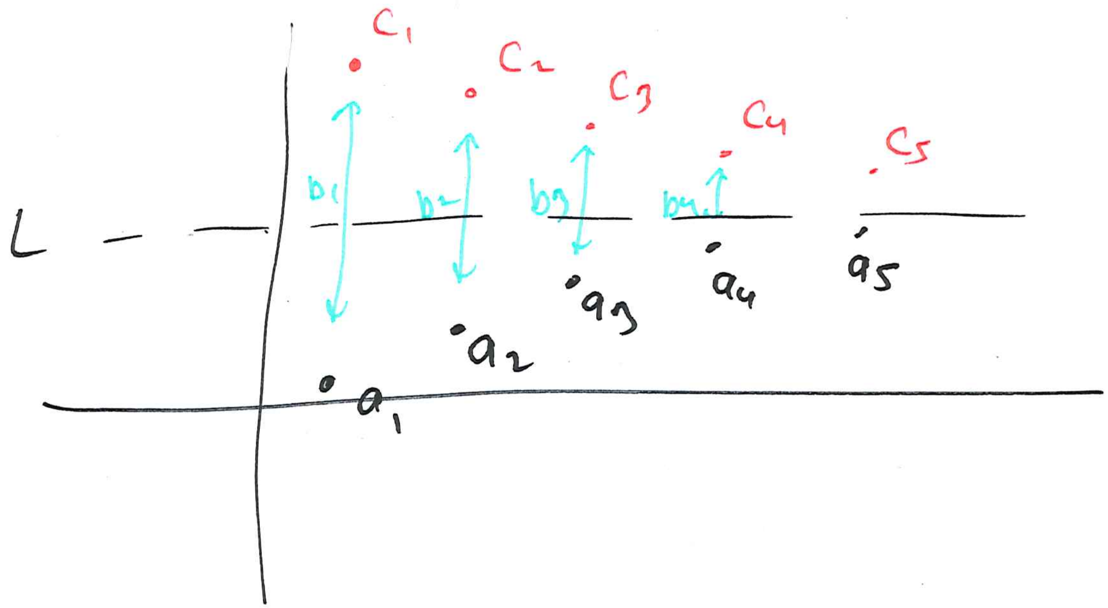
$\lim_{n\to\infty} a_n b_n = 0 \cdot \infty$

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} \left(\frac{1}{n^2} \cdot 2n^2\right) = \lim_{n\to\infty} 2 = 2$$

Squeeze Thm for Limits of Sequences

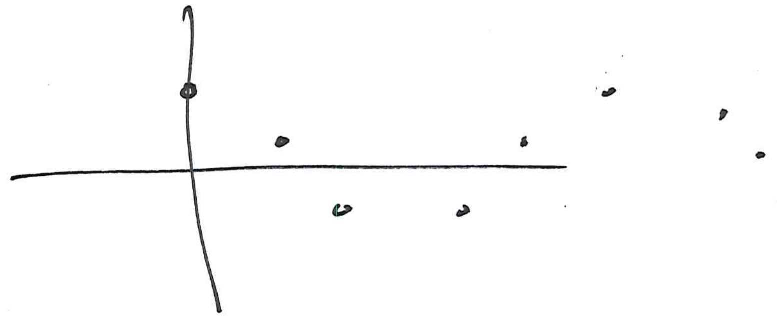
Let $\{a_n\}$, $\{b_n\}$ $\{c_n\}$ be sequences with $a_n \leq b_n \leq c_n$ for all integers n greater than some index N.

If $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ as well.



We use this usually when other limit laws don't apply because a limit DNE (ex) $$\lim_{n\to\infty} \frac{2n+\cos n}{n+1}$$

Note: $\lim_{n\to\infty} \cos n$ DNE



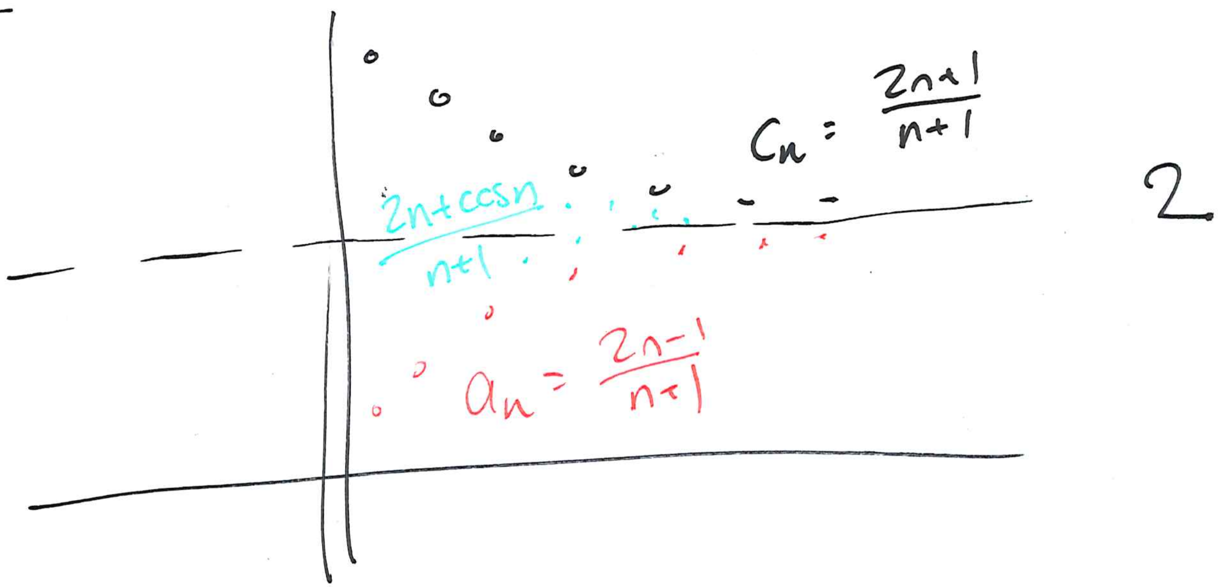
Squeeze Thm:

$$-1 \le \cos n \le 1$$

So: $$\underbrace{\frac{2n-1}{n+1}}_{a_n} \le \frac{2n+\cos n}{n+1} \le \underbrace{\frac{2n+1}{n+1}}_{c_n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{2n-1}{n+1}\right) = \frac{2}{1} = 2$$

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \left(\frac{2n+1}{n+1}\right) = \frac{2}{1} = 2$$



- $\dfrac{2n-1}{n+1} \leq \dfrac{2n+\cos n}{n+1} \leq \dfrac{2n+1}{n+1}$

- $\lim\limits_{n\to\infty}\left(\dfrac{2n-1}{n+1}\right) = \lim\limits_{n\to\infty}\left(\dfrac{2n+1}{n+1}\right) = 2$

So, by Squeeze Theorem, also

$$\lim_{n\to\infty}\left(\frac{2n+\cos n}{n+1}\right) = 2$$

To write up: match wording of theorem

- show "floor" "ceiling" sequences sequence you care about in between
- show "floor" "ceiling" sequences have same limit
- conclude: your sequence has that limit as well.