Remember from Last Time:

Let X be a continuous random variable.

Its cumulative distribution function is $F(x) = Pr(X \leq x)$

eg $F(20)$ is the probability X is at most 20

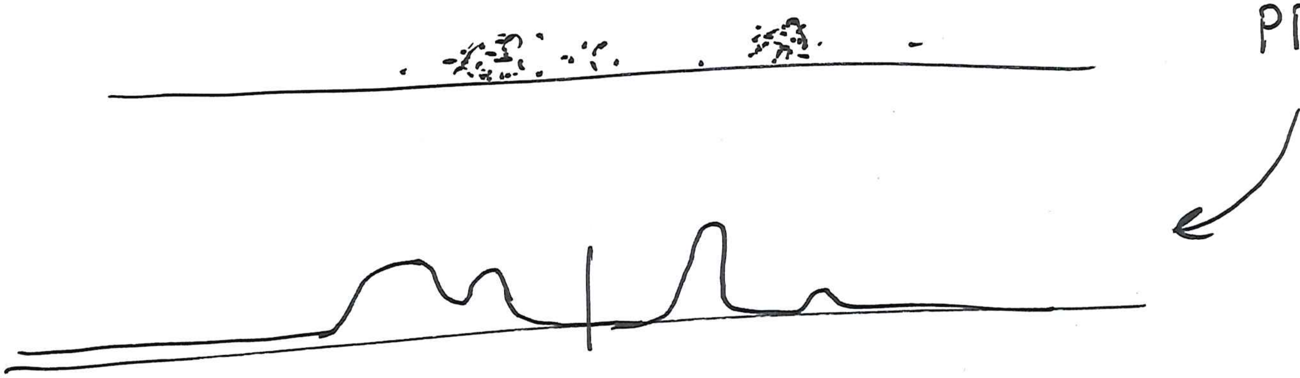
Its probability density function is $f(x) = F'(x)$.

This is a measure of which regions are more or less likely to be near X.

It is scaled so that:

$$Pr(a \leq X \leq b) = \int_a^b f(x)\,dx = F(x).$$

We use the PDF because $Pr(X = x) = 0$ for all values x: the odds of X being any one number are infintesimally small.



PDF:

Properties of Probability Density Functions

$$F(x) \stackrel{\text{def}}{=} Pr(X \le x) = Pr(-\infty < X \le x)$$

$$= \int_{-\infty}^{x} f(t)\,dt$$

1. CDF $F(x) = \int_{-\infty}^{x} f(t)\,dt$

PDF

why did x change to t?

$$F(X) = \int_{-\infty}^{X} f(x)\,dx$$

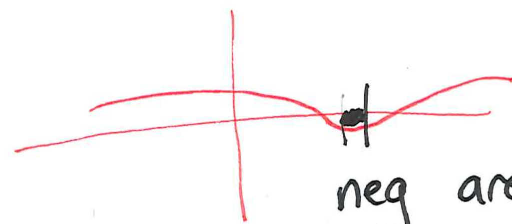
$$F(0) = \int_{-\infty}^{0} f(0)\,d0$$

???

$$\int_a^b f(x)\,dx = Pr(a \le X \le b) \ge 0$$

$\Rightarrow f(x) \ge 0$

If $f(x) < 0$:



neg area $\rightarrow$ neg probability (nonsense)

② $f(x) \ge 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = Pr(-\infty \le X \le \infty) = 1$$

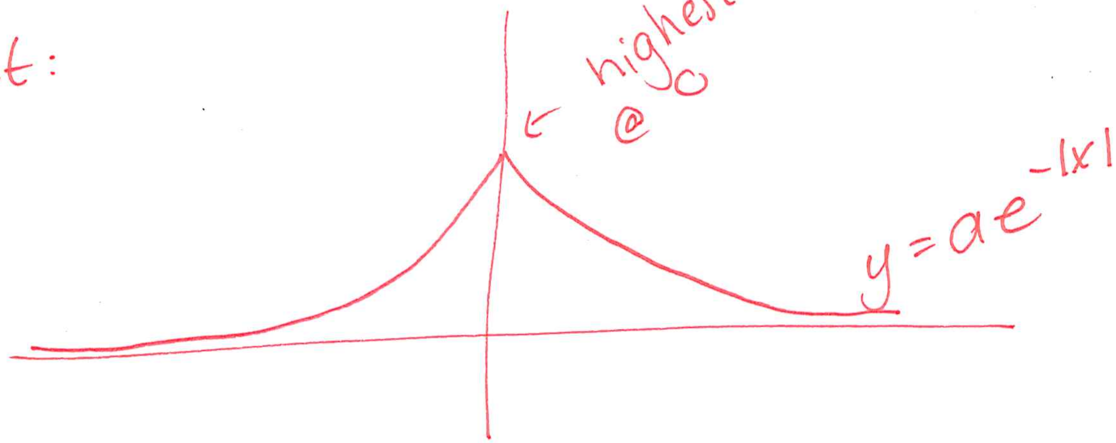
<u>all</u> values are in this range

③ $\int_{-\infty}^{\infty} f(x)\,dx = 1$

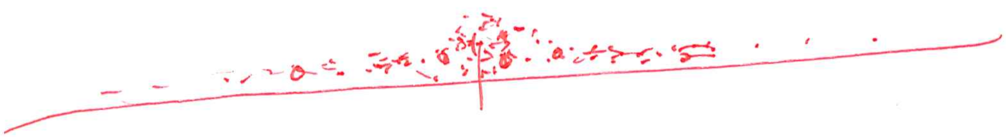
(ex) Let's say $f(x) = ae^{-|x|}$ is a probability density function

What is a? (constant)

Interpret:



Values are clustered near 0



$\Pr(-\infty \le X \le \infty)$ 100%

Property 3:

$$\int_{-\infty}^{\infty} ae^{-|x|}\,dx = 1$$

Note: If $x \geq 0$ then $|x| = x$

If $x < 0$ then $\underbrace{|x|}_{\text{changes sign of } x} = -x$

(neg x)(neg 1) = pos

$$1 = \int_{-\infty}^{\infty} ae^{-|x|}\,dx =$$

$$\int_{-\infty}^{0} ae^{x}\,dx + \underbrace{\int_{0}^{\infty} ae^{-x}\,dx}_{x \geq 0}$$

So $|x| = x$

So $-|x| = -x$

$$\lim_{s \to -\infty} \int_{s}^{0} ae^{x}\,dx + \lim_{r \to \infty} \int_{0}^{r} ae^{-x}\,dx$$

$$\lim_{s \to -\infty} \left(\underbrace{ae^{0}}_{a} - \underbrace{ae^{s}}_{\to 0}\right) + \lim_{r \to \infty} \left(\underbrace{-ae^{-r}}_{\to 0} \underbrace{+ (+a)e^{-0}}_{a}\right)$$

$$= 2a = 1$$

$$\boxed{\text{So: } a = 1/2}$$

$$e^{-r} = \frac{1}{e^{r}}$$

Expected Value

discrete ex: 3 students 70%
2 students 50%

Average: $$\frac{70+70+70 \quad + \quad 50+50}{5}$$

$$= \frac{3 \cdot 70 + 2 \cdot 50}{5}$$

$$= \frac{3}{5} 70 + \frac{2}{5} 50$$

proportion of students scoring 70

proportion scoring 50

Avg: weighted sum of possible values
↳ multiply by likelihood

For a continuous random variable X with probability density function $f(x)$, the <u>expected value</u> ("mean" or "expectation") is

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} \underset{\text{value}}{x} \; \underset{\text{likelihood}}{f(x)} \, dx$$

memorize def

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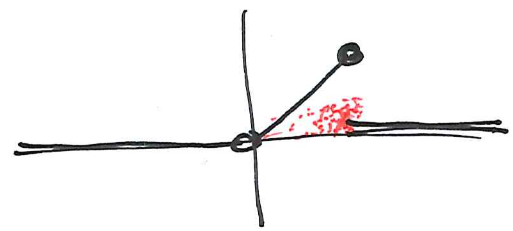
ex

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



PDF

Imagining finding lots of these values, and averaging.

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx = \underbrace{\int_{-\infty}^{0} x \underbrace{f(x)}_{0} dx}_{0} + \int_0^1 x \underbrace{f(x)}_{2x} dx + \underbrace{\int_1^{\infty} x \underbrace{f(x)}_{0} dx}_{0}$$

$$= \int_0^1 x(2x)\, dx = \int_0^1 2x^2\, dx = \frac{2}{3} x^3 \Big|_0^1 = \boxed{2/3}$$

Do a million trials, <u>average</u> $\approx 2/3$

Does this make sense?

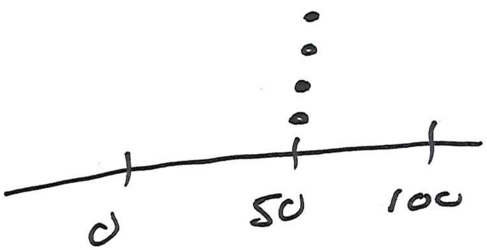
All values in $[0,1] \longrightarrow$ avg in $[0,1]$

Cluster closer to 1 than to $0 \rightarrow$ avg $> 1/2$

(more big # than small #s)

Variance + Standard Deviation

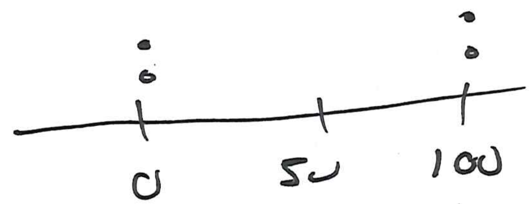
Motivation



Exam 1

Avg : 50

Avg described each individual well)



Exam 2

Avg: $\frac{1}{2}(0) + \frac{1}{2}(100)$

$= 50$

Avg describes individuals poorly

Idea: avg (difference between individual and average)

$X - \mathbb{E}(X)$

quantity I want to average

value of individual

avg

$(X - \mathbb{E}(X))^2$

make everything positive (or 0)

$$\boxed{\int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 \cdot f(x)\, dx}$$

memorize

Defines Variance of X, written Var(X)

FACT: also

$$\mathrm{Var}(X) = \mathbb{E}(X^2) + [\mathbb{E}(X)]^2$$

alternate computation - gives same value

Def: $\underline{\text{Standard deviation}}$ of X is

$$\sigma(X) = \sqrt{\mathrm{Var}(X)}$$

sigma

to scale, after we squared things

ex $f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{else} \end{cases}$ is a PDF of X

We found ~~$\mathbb{E}(2/3)$~~ $\mathbb{E}(X) = 2/3$

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 f(x)\, dx$$

$$= \underbrace{\int_{-\infty}^{0} (x-\mathbb{E}(X))^2 \underbrace{f(x)}_{0} dx}_{0} + \int_0^1 (x - \underbrace{\mathbb{E}(X)}_{2/3})^2 \underbrace{f(x)}_{2x} dx + \underbrace{\int_1^{\infty} [x-\mathbb{E}(X)]^2 \underbrace{f(x)}_{0} dx}_{0}$$

$$= \int_0^1 (x-\mathbb{E}(X))^2 \cdot 2x\, dx = \int_0^1 (x - 2/3)^2\, 2x\, dx$$

$$= \int_0^1 (x^2 - 4/3\,x + 4/9)\, 2x\, dx = \int_0^1 \left[2x^3 - \frac{8}{3}x^2 + \frac{8}{9}x\right] dx$$

$$= \frac{2}{4} - \frac{8}{9} + \frac{4}{9} = \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$$

Avg dist from $2/3$ (squared) is $\frac{1}{18}$

Other way to compute Var(X):

$$\underbrace{\mathbb{E}(X^2)}_{\text{find}} - [\underbrace{\mathbb{E}(X)}_{2/3}]^2$$

def of $\mathbb{E}$: $\int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx =$

$$\int_0^1 x^2(2x)\,dx = \int_0^1 2x^3\,dx$$

$$= \frac{2}{4}x^4\Big|_0^1 = \frac{1}{2} = \mathbb{E}(X^2)$$

$$Var(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{1}{2} - \left(\frac{2}{3}\right)^2$$

$$= \frac{1}{2} - \frac{4}{9} = \frac{9-8}{18} = \frac{1}{18}$$

Standard Deviation: $\sigma(X) = \sqrt{Var(X)}$

$$= \frac{1}{\sqrt{18}} = \frac{1}{3\sqrt{2}}$$

$\sigma(X) = \sqrt{\text{Var}X}$

What if $\text{Var}(X) < 0$?

Fact: $\text{Var}(X)$ never negative