Ch 7.9 : Differential Equations

A differential equation involves the derivative of a function

e.g. $\frac{dy}{dx} = yx$

Antiderivatives are a (simple) kind of diff eq

(ex) $y(t)$ is a function with $y(0)=1$, $y(1)=10$,
and $y''(t) = 12t+1$
what is $y(t)$?

$$y'(t) = 6t^2 + t + C$$
$$y(t) = 2t^3 + \tfrac{1}{2}t^2 + Ct + D$$

$y(0)=1$:

$$1 = 0 + 0 + 0 + D$$
$$D=1$$

$y(1)=10$:

$$y(t) = 2t^3 + \tfrac{1}{2}t^2 + Ct + 1$$
$$10 = 2 + \tfrac{1}{2} + C + 1$$
$$6.5 = C$$

So:

$$\boxed{y(t) = 2t^3 + \tfrac{1}{2}t^2 + \tfrac{13}{2}t + 1}$$

Before we <u>solve</u> more diff eqs, let's practice understanding notation.

(ex) $\frac{dy}{dx} + x^2 - 1 = y$

which of the following are solutions:

(A) $y = x^2 + 1$

(B) $y = x^2 + 2x + 1$

(C) $y = -\frac{1}{3}x^3 + x$

(A) $y = x^2 + 1$

So $\frac{dy}{dx} + x^2 - 1 = y$

means

$2x + x^2 - 1 = x^2 + 1$

These fcns not same!

So $y = x^2 + 1$ is <u>not</u> a solution

(B) $(y = x^2 + 2x + 1)$

$\frac{dy}{dx} + x^2 - 1 = y$ means

$(2x+2) + x^2 - 1 = x^2 + 2x + 1$

$x^2 + 2x + 1 = x^2 + 2x + 1$

True! So $y = x^2 + 2x + 1$ is a solution

(c) If $y = \frac{-1}{3}x^3 + x$, then:

$$\frac{dy}{dx} + x^2 - 1 = y$$

$$(-x^2+1) + x^2 - 1 = \frac{-1}{3}x^3 + x$$

$$\underbrace{0}_{} = \underbrace{\frac{-1}{3}x^3 + x}_{}$$

Not same function

So $y = \frac{-1}{3}x^3 + x$ not a solution.

(ex) $\frac{dy}{dx} = \frac{x}{y}$

Which are a solution?

(A) $y = -x$ (B) $y = x + 5$

(C) $y = \sqrt{x^2+5}$

(A) $\frac{dy}{dx} = -1$ $\qquad \frac{x}{y} = \frac{x}{-x} = -1$

So: $\frac{dy}{dx} = \frac{x}{y}$ $\qquad y = -x$ is a solution

(B) $\frac{dy}{dx} = 1$ $\qquad \frac{x}{y} = \frac{x}{x+5} \neq 1$ $\qquad$ So $y = x+5$ is not a solution

(C) $\frac{dy}{dx} = \frac{2x}{2\sqrt{(x^2+5)}} = \frac{x}{\sqrt{x^2+5}} = \frac{x}{y}$ $\qquad$ So $y = \sqrt{x^2+5}$ is a solution

Separable Differential Equations

Idea: $$g(y)\cdot\frac{dy}{dx} = f(x)$$

$$\int g(y)\frac{dy}{dx}\,dx = \int f(x)\,dx$$

This is what we do in a u-sub (we call our function y instead of u)

$$\int g(y)\,dy = \int f(x)\,dx$$

no x's in here

y is my variable of integration

Shorthand:

$$g(y)\frac{dy}{dx} = f(x)$$

- mult both sides by dx
- integrate

$$\int g(y)\,dy = \int f(x)\,dx$$

ex $$\frac{dy}{dx} = y^2x$$

Separate variables

$$\frac{1}{y^2}\frac{dy}{dx} = x$$

$$\frac{1}{y^2}dy = x\,dx$$

$$\int\frac{1}{y^2}dy = \int x\,dx$$

$$-\frac{1}{y} = \frac{1}{2}x^2 + C$$

"implicit" not: $y = \dots$

$$\frac{y}{-1} = \frac{1}{\frac{1}{2}x^2+C}$$

$$\boxed{y = \frac{-1}{\frac{1}{2}x^2+C}}$$

"explicit"

$f(x) + C_1 = g(x) + C_2$ $\quad$ C_1, C_2 any constats

$\Rightarrow$ $f(x) = g(x) + (C_2 - C_1)$

Set $C = C_2 - C_1$

$f(x) = g(x) + C$

That's why we can get away with <u>one</u> constant c instead of 2 ex $\frac{dy}{dx} = x\sec y$ and $y(0)=1$

① separate variables

$$\frac{1}{\sec y}\frac{dy}{dx} = x$$

$$\frac{1}{\sec y}\,dy = x\,dx$$

② integrate

$$\int \frac{1}{\sec y}\,dy = \int x\,dx$$

$$\int \cos y\,dy = \frac{1}{2}x^2 + C$$

$$\sin y = \frac{1}{2}x^2 + C$$

implicit

③ Find C

If $y(0)=1$:

$$\sin(1) = \frac{1}{2}0^2 + C$$

$$\boxed{\sin 1 = C}$$

So:

$$\boxed{\sin y = \frac{1}{2}x^2 + \sin 1}$$

implicit

④ Find explicit formula:

$$\boxed{y = \arcsin\left(\frac{1}{2}x^2 + \sin 1\right)}$$

ex $\frac{dy}{dx} = y(4x^3-1)$ and $y(0) = -2$

Find an explicit formula for y.

$\int \frac{1}{y}\,dy = \int (4x^3-1)\,dx$ $\left(\begin{array}{c} \cdot dx \\ \div y \end{array} \right)$

If y is a function of x, it passes vertical line test

i.e. for any x in domain, only one y

$\ln|y| = x^4 - x + C$

If $x=0$, $y=-2$

$\ln|-2| = 0^4 - 0 + C$

$\ln 2 = C$

So: $\ln|y| = x^4 - x + \ln 2$

$|y| = e^{x^4-x+\ln 2} = e^{x^4-x}e^{\ln 2} = 2e^{x^4-x}$

This describes two possible functions:

$y = 2e^{x^4-x}$

$y = -2e^{x^4-x}$

We need to choose

1st case: $y(0) = 2e^0 = 2$ ✗

2nd case: $y(0) = -2e^0 = -2$ ✓

$$\boxed{y = -2e^{x^4-x}}$$

1st-order diff eq: involves y' (no y'' etc)

eg $y' = y$

$y' + 1 = y$

2nd-order diff eq: involve y'' (maybe y') (not y''' etc)

eg $y'' = y$

$y'' + y' + y = 0$

Linear first-order diff eq:

$$y' = \underbrace{ky}_{\text{linear in } y} + b$$

Really common!

ex $\quad \dfrac{dy}{dt} = ky + b = k(y + b/k)$

$\displaystyle\int \frac{1}{y+b/k}\, dy = \int k\, dt$

$\ln|y + b/k| = kt + C$ $\qquad$ C - constant

$|y + b/k| = e^{kt} e^{c}$ $\qquad$ e^c - ~~ans~~ another constant

might as well write C instead of e^c

$|y + b/k| = C e^{kt}$

$y + \frac{b}{k} = \pm C e^{kt}$ $\qquad$ $\pm C$: just another constant

$y + \frac{b}{k} = C e^{kt}$

$\boxed{y = Ce^{kt} - b/k}$ solutions to $y' = ky + b$

(memorize)

ex Solve $y' = 3y + 7$, $y(2) = 5$

$k = 3$ $b = 7$

Plug in: $y = Ce^{3t} - 7/3$ Find C

$5 = Ce^{3\cdot 2} - 7/3$

$\frac{15}{3} + \frac{7}{3} = Ce^{6}$

$\frac{22}{3} = Ce^{6}$

$C = \frac{22}{3e^{6}}$

So: $\boxed{y = \frac{22}{3e^{6}} e^{3t} - 7/3}$

When we solve a diff eq, we can check (like multiple choice at start of class)

$$y' = 3y + 7 \qquad \text{If } y = \frac{22}{3e^6}e^{3t} - 7/3 :$$

$$\left(\left(\frac{22}{3e^6}\right)e^{3t}\cdot 3\right) \stackrel{?}{=} 3\underbrace{\left(\frac{22}{3e^6}e^{3t} - 7/3\right)}_{y} + 7$$

$$\frac{22}{e^6}e^{3t} \stackrel{?}{=} \frac{22}{e^6}e^{3t} \cancel{-7} \cancel{+7}$$

TRUE: so our solution is correct

ex $y' = 2y + 8$:

$k = 2$ $b = 8$

Then $y = Ce^{kt} - b/k$ ← If you forget this, solve using separation of variables.

$$\boxed{y = Ce^{2t} - 4}$$