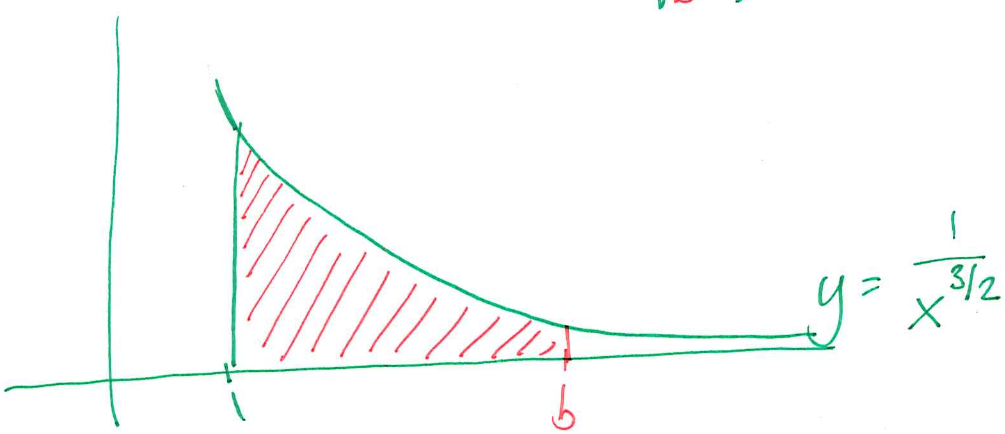
Recall from Last Time:

We deal with improper integrals using limits.

ex)
$$\int_1^\infty \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^{3/2}}\,dx$$

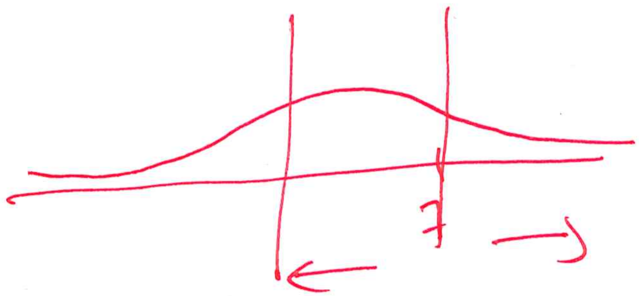
$$= \lim_{b\to\infty}\left(-2\frac{1}{\sqrt{x}}\Big|_1^b\right) = \lim_{b\to\infty}\left(\frac{-2}{\sqrt{b}} - \frac{-2}{\sqrt{1}}\right)$$

$$= \lim_{b\to\infty}\left(2 - \frac{2}{\sqrt{b}}\right) = 2 - 0 = 2$$



As we move b farther and farther to the right, the shaded area gets closer and closer to 2 units.

(ex) $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{7} \frac{1}{1+x^2}\,dx + \int_{7}^{\infty} \frac{1}{1+x^2}\,dx$$



(doesn't matter where we break it)

$$= \lim_{a\to-\infty} \underbrace{\int_a^7 \frac{1}{1+x^2}\,dx}_{\text{finite}} + \lim_{b\to\infty} \underbrace{\int_7^b \frac{1}{1+x^2}\,dx}_{\text{finite}}$$

$$= \lim_{a\to-\infty}\left(\underbrace{\arctan 7}_{\#} - \underbrace{\arctan a}_{-\pi/2}\right) + \lim_{b\to\infty}\left(\underbrace{\arctan b}_{\pi/2} - \underbrace{\arctan 7}_{\#}\right)$$

$$= \cancel{\arctan 7} + \pi/2 + \pi/2 - \cancel{\arctan 7} = \pi$$

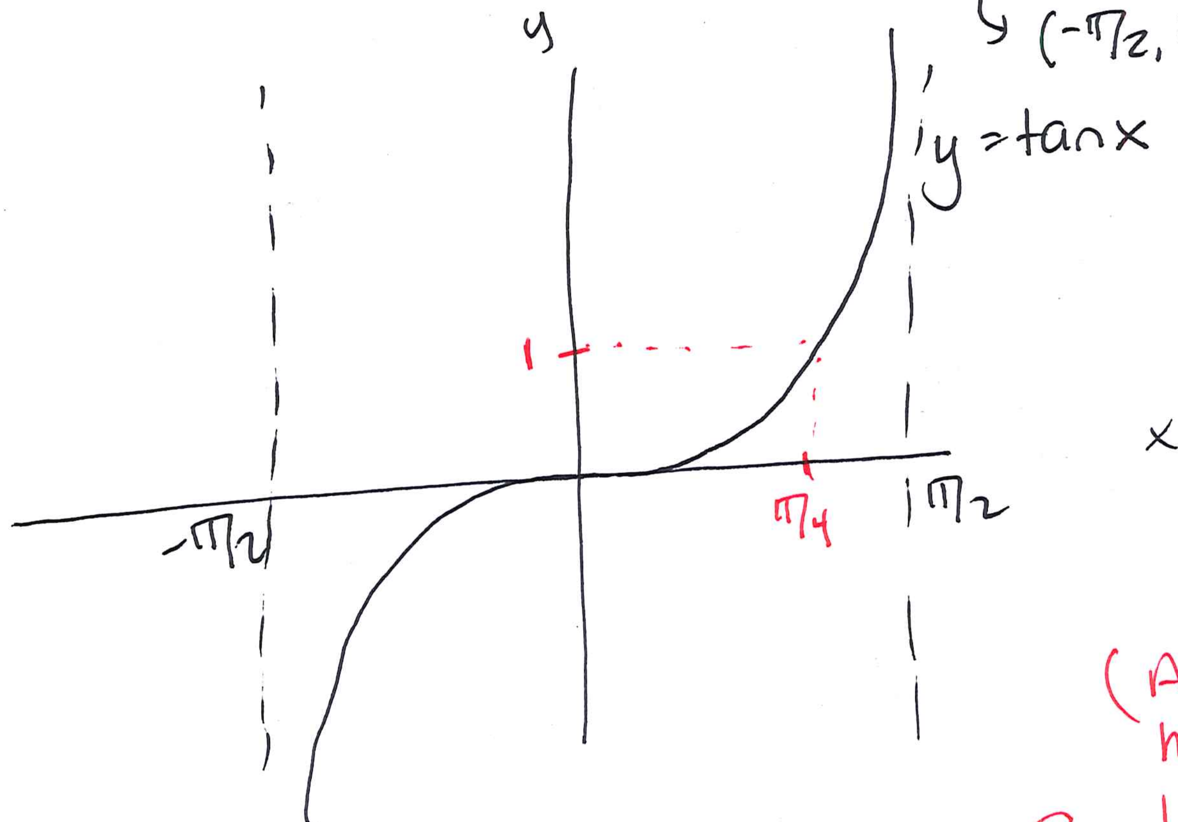
arctan is inv function of tangent

$\tan(\pi/4) = 1$ so $\arctan 1 = \pi/4$

$\tan(0) = 0$ so $\arctan(0) = 0$

"arctan x" means:

which angle gives tan of x? ↳ $(-\pi/2, \pi/2)$



$\arctan(100) \approx \pi/2$

$\lim_{x \to \infty} \arctan(x) = \pi/2$

$\arctan(-100) \approx -\pi/2$

(Angles close to $-\pi/2$ have hugely negative tangents)

So $\lim_{x \to -\infty} \arctan x = -\pi/2$

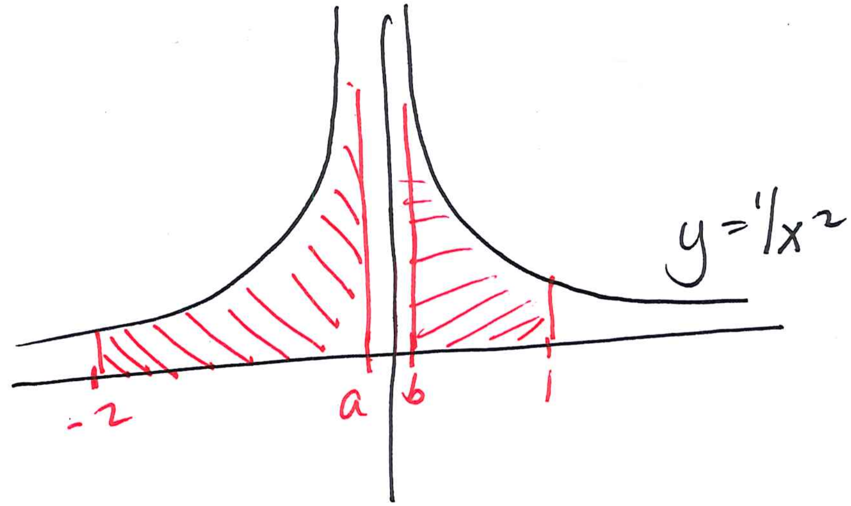
ex $\displaystyle\int_{-2}^{1} \frac{1}{x^2}\,dx$

Finite region of integration
$[-2, 1]$

$\frac{1}{x^2}$ not bounded in this region – infinite discontinuity

$$= \int_{-2}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{a\to 0^-} \int_{-2}^{a} \frac{1}{x^2}\,dx + \lim_{b\to 0^+} \underbrace{\int_{b}^{1} \frac{1}{x^2}\,dx}_{\text{finite}}$$



$$= \lim_{a\to 0^-}\left[\frac{-1}{x}\Big|_{-2}^{a} \right] + \lim_{b\to 0^+}\left[\frac{-1}{x}\Big|_{b}^{1} \right]$$

We'll replace problematic piece with limits

If one limit DNE (∞) we say the entire integral diverges

DIV

$$= \lim_{a\to 0}\Big[\underbrace{\frac{-1}{a}}_{\infty} - \frac{-1}{-2} \Big] + \lim_{b\to 0^+}\left[\frac{-1}{1} - \frac{-1}{b} \right]$$

$= \infty + (\text{something})$

So: integral diverges

Q: Do $+\infty$ & $-\infty$ cancel out?

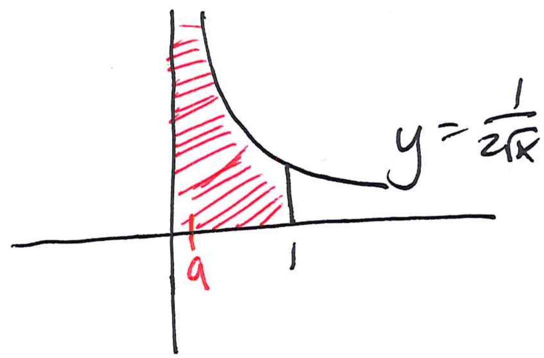
A: NO

eg:

$$2\infty = \infty$$
$$-\infty \qquad -\infty$$
$$\infty = 0$$

∞ not a number!

(ex) $\displaystyle\int_0^1 \frac{1}{2\sqrt{x}}\,dx$



$$= \lim_{a\to 0^+} \int_a^1 \frac{1}{2\sqrt{x}}\,dx$$

$$= \lim_{a\to 0^+} \left[\sqrt{1} - \sqrt{a}\right]$$

$$= 1 - 0 = \boxed{1}$$

In calculus, we solve our problems by ignoring them (temporarily)

FACTS:

$\int_0^1 \frac{1}{\sqrt{x}}\,dx$ CON

$\int_0^1 \frac{1}{x}\,dx$ DIV

$\int_0^1 \frac{1}{x^2}\,dx$ DIV

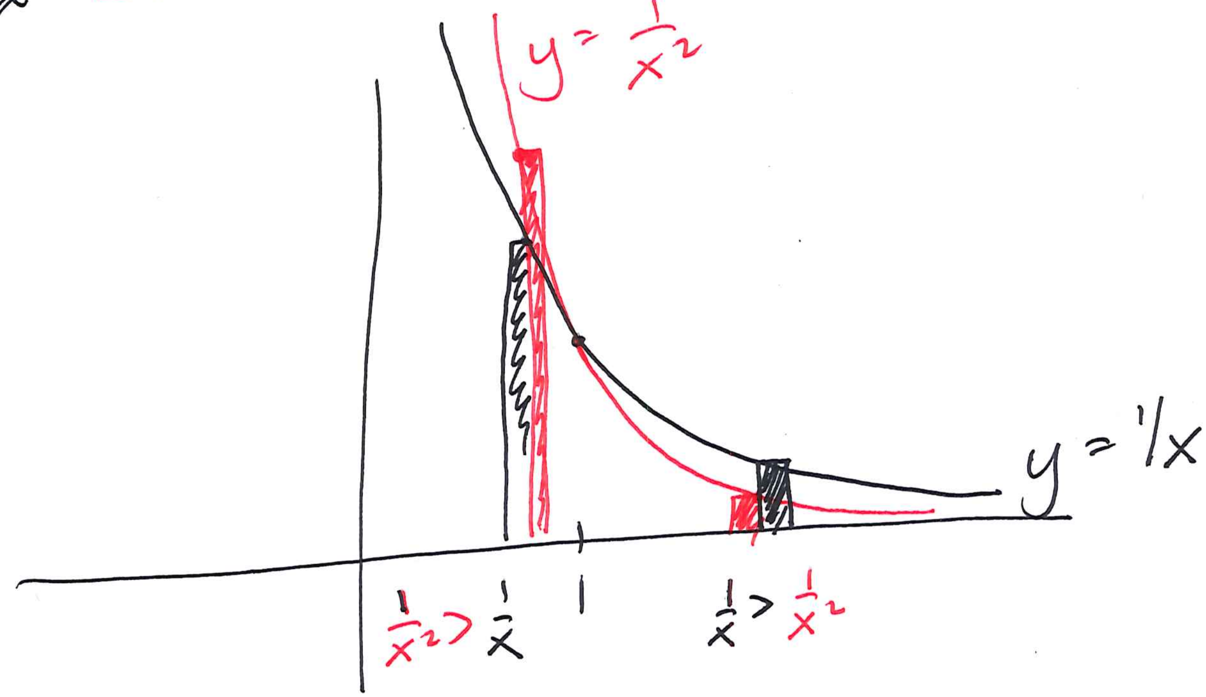
$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ DIV

$\int_1^\infty \frac{1}{x}\,dx$ DIV

$\int_1^\infty \frac{1}{x^2}\,dx$ CONV

If $x<1$, $x^2 < x$

If $x>1$, $x^2 > x$



p-test ("Power")

p: constant

$$\int_0^1 \frac{1}{x^p}\,dx \qquad \text{CONV IF } p < 1 \quad \text{DIV IF } p \geq 1$$

$$\int_1^\infty \frac{1}{x^p}\,dx \qquad \text{CONV IF } p > 1 \quad \text{DIV IF } p \leq 1$$

eg $\int_0^2 \frac{1}{x^{1/3}}\,dx$ $\quad p = 1/3 < 1$

so: CONV by p-test

eg $\int_5^\infty \frac{1}{x^{0.999}}\,dx$ $\quad p = 0.999 < 1$

so: DIV by p-test

ex $\displaystyle\int_e^\infty \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \underbrace{\int_e^b \frac{1}{x\ln x}\,dx}$

$u = \ln x$

$du = \frac{1}{x}dx$

If $x=b$:

$u = \ln x = \ln b$

If $x=e$:

$u = \ln x = \ln e = 1$

$\displaystyle = \lim_{b\to\infty}\left[\int_1^{\ln b} \frac{1}{u}\,du\right]$

$\displaystyle = \lim_{b\to\infty}\left[\ln|\ln b| - \underbrace{\ln 1}_{0}\right]$

$\displaystyle = \lim_{b\to\infty} \underbrace{\ln|\underbrace{\ln b}_{\to\infty}|}_{\to\infty} = \infty$

Integral diverges.

Ch 7.9 Differential Equations

Let's get used to these equations before we learn to solve them.

eg. $\frac{dy}{dx} + x^2 - 1 = y$

There's some fcn $y(x)$ that makes this true

Which are solutions:

Ⓐ $y = x^2 + 1$

Ⓑ $y = x^2 + 2x + 1$

Ⓒ $y = -\frac{1}{3}x^3 + x$

Ⓐ If $y = x^2 + 1$

then $\frac{dy}{dx} + x^2 - 1 = y$

means $2x + x^2 - 1 = x^2 + 1$

Diff functions, so FALSE

so $y = x^2 + 1$ NOT a solution

Ⓑ If $y = x^2 + 2x + 1$

then $\frac{dy}{dx} + x^2 - 1 = y$

means $(2x + 2) + x^2 - 1 = (x^2 + 2x + 1)$

ie $x^2 + 2x + 1 = x^2 + 2x + 1$

TRUE: so $y = x^2 + 2x + 1$ IS a solution eg $\frac{dy}{dx} = \frac{x}{y}$

Which are solutions?

✓ (A) $y = -x$

✗ (B) $y = x+5$

✓ (C) $y = \sqrt{x^2+5}$

(A) If $y = -x$:

$\frac{dy}{dx} = \frac{x}{y}$

means: $-1 = \frac{x}{-x}$

true

(B) If $y = x+5$

$\frac{dy}{dx} = \frac{x}{y}$

$\Leftrightarrow$ $1 = \frac{x}{x+5}$ FALSE

not a soln

(C) If $y = \sqrt{x^2+5}$

$\frac{dy}{dx} = \frac{x}{y}$ means:

$\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+5}} = \frac{x}{\sqrt{x^2+5}}$ TRUE

Antidifferentiation solves a (simple) kind of diff eq:

(ex) $\frac{dy}{dx} = e^x$: what is y?

$y = \int e^x dx = e^x + C$

what if $y(0) = 2$?
ie when $x=0$, $y=2$

$y = e^x + C$

$2 = e^0 + C = 1 + C$

so $C = 1$

So: $\boxed{y = e^x + 1}$

Only 1st deriv:
"first-order differential equation"

(ex) $y''(t) = 12t + 1$ "second-order diff eq"

and $y(0) = 1$ and $y(1) = 10$

Find y.

$y'(t) = 6t^2 + t + C$

So $y(t) = 2t^3 + \frac{1}{2}t^2 + Ct + D$

$1 = 2 \cdot 0^3 + \frac{1}{2}0^2 + C \cdot 0 + D$ $(y(0) = 1)$

$1 = D$

$10 = 2 + \frac{1}{2} + C + D$ $(y(1) = 10)$

$10 = 2.5 + C + 1$

$10 = 3.5 + C$

$6.5 = C$

$$\boxed{y(t) = 2t^3 + \frac{1}{2}t^2 + 6.5t + 1}$$

Equations of the form

$$\frac{dy}{dx} = ky + b$$

"linear first-order diff eq"

where k, b constants

FACT: All solutions have the form

$$y(x) = Ce^{kx} - b/k$$

(we won't prove this - memorize)

ⓔⓧ $y' = 3y + 7$, $y(2) = 5$ What is y?

$k = 3$ $b = 7$

So: $y = Ce^{3x} - 7/3$

(plugging in)

$$5 = Ce^{3 \cdot 2} - 7/3$$

$$\frac{22}{3} = Ce^6$$

$$\frac{22}{3e^6} = C$$

So: $\boxed{y = \frac{22}{3e^6} e^{3x} - 7/3}$

Check: If $y = \frac{22}{3e^6} e^{3x} - 7/3$

then $y' = 3y + 7$

means $\underbrace{\frac{22}{3e^6} \cdot e^{3x} \cdot 3}_{y'} = 3 \cdot \underbrace{\left(\frac{22}{3e^6} e^{3x} - 7/3 \right)}_{y} + 7$

$$3 \cdot \frac{22}{3e^6} e^{3x} = 3 \cdot \frac{22}{3e^6} e^{3x} \cancel{-7} \cancel{+7}$$

TRUE

(ex) $y' = 2y + 8, \quad y(0) = 3$ What is y?

(FACT: If $y' = ky + b$, then $y = Ce^{kx} - b/k$)

$k \to 2$, $b \to 8$

$y = Ce^{2x} - 8/2$

$y = Ce^{2x} - 4$

$3 = Ce^{2 \cdot 0} - 4 = C - 4$

So $C = 7$

(plugging in)

$y = 7e^{2x} - 4$

Separable Differential Equations
(best part of Ch 7.9)

Start with: $g(y) \cdot \frac{dy}{dx} = f(x)$

Then: $\int \left(g(y) \cdot \frac{dy}{\cancel{dx}} \right) \cancel{dx} = \int f(x)\,dx + C$

Like u-sub
(think of "y" as "u")

$$\int g(y)\,dy = \int f(x)\,dx + C$$

Shorthand: $g(y) \frac{dy}{dx} = f(x)$

$\Rightarrow \quad g(y)\,dy = f(x)\,dx$

$\Rightarrow \quad \int g(y)\,dy = \int f(x)\,dx$

ex

$$\frac{dy}{dx} = y^2 x \quad ,$$

① "Separate" y's from x's

$$\frac{1}{y^2}\frac{dy}{dx} = x$$

$$\underbrace{\frac{1}{y^2}\,dy}_{\text{only } y\text{'s}} = \underbrace{x\,dx}_{\text{only } x\text{'s}}$$

① Integrate

$$\int \frac{1}{y^2}\,dy = \int x\,dx$$

$$\frac{-1}{y} = \frac{1}{2}x^2 + C$$

$$\frac{1}{y} = -\frac{1}{2}x^2 - C$$

$$\boxed{y = \frac{1}{-\frac{1}{2}x^2 - C}}$$

$$\int a\,dy = \int b\,dx$$

$$ay + C_1 = bx + C_2$$

$$ay = bx + \underbrace{(C_2 - C_1)}_{C}$$

$$ay = bx + C$$

"implicit" function of y
not "y = "

"explicit" form
y = ...

ex $\dfrac{dy}{dx} = x \sec y$

① Separate vars ($\cdot$, $\div$)

$$\frac{1}{\sec y} \cdot \frac{dy}{dx} = x$$

$$\cos y \cdot \frac{dy}{dx} = x$$

$$\cos y \cdot dy = x\, dx$$

$$\int \cos y\, dy = \int x\, dx$$

$\sin y = \frac{1}{2}x^2 + C$ (implicit)

$\boxed{y = \arcsin\left(\frac{1}{2}x^2 + C\right)}$ (explicit)

Say $y(0) = 1$:

$$\sin y = \tfrac{1}{2}x^2 + C$$

$$\sin(1) = \tfrac{1}{2}0^2 + C$$

$$\sin 1 = C$$

So:

$$\boxed{y = \arcsin\left(\tfrac{1}{2}x^2 + \sin 1\right)}$$

ex $\frac{dy}{dx} = y(4x^3-1)$, $y(0) = -2$

mult dx both sides
div y both sides
Integral sign

$$\int \frac{1}{y}\,dy = \int (4x^3-1)\,dx$$

$$\ln|y| = x^4 - x + C$$

$$\ln|-2| = 0^4 - 0 + C$$

$$\ln 2 = C$$

So: $\boxed{\ln|y| = x^4 - x + \ln 2}$ (implicit)

When we write explicit form we want y to be a function (pass vertical line test: one input never gives 2 outputs)

$$|y| = e^{x^4 - x + \ln 2} = e^{x^4-x}e^{\ln 2}$$

$$= 2e^{x^4-x}$$

could be: $y = \underbrace{2e^{x^4-x}}_{\text{pos}}$ $y(0)=+2$

or could be: $|y| = -y$ (y neg) $y = \underbrace{-2e^{x^4-x}}_{\text{neg}}$

To choose look to initial conditions $y(0) = -2$

So: $\boxed{y = -2e^{x^4-x}}$