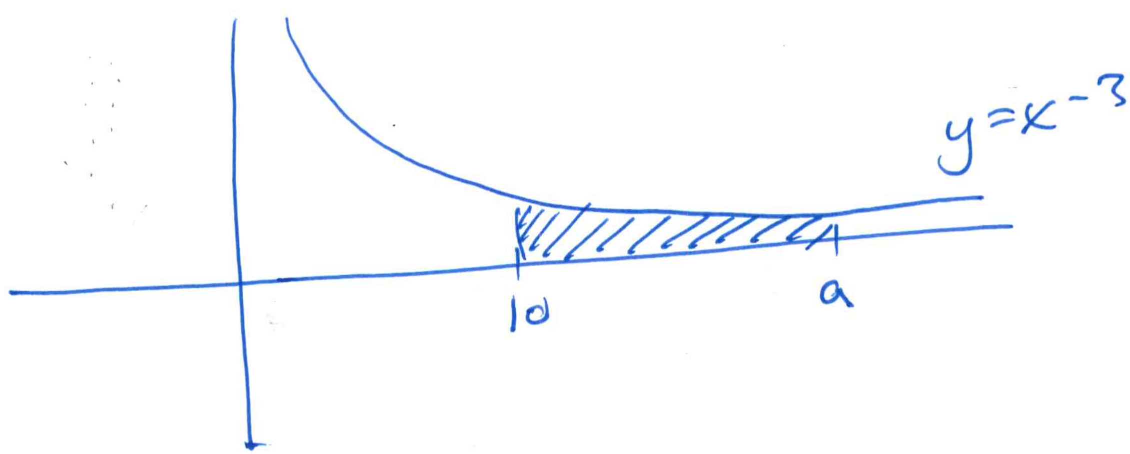
Remember from Last Time:

We interpret improper integrals using limits.

ex
$$\int_{10}^{\infty} x^{-3}\,dx = \lim_{a\to\infty} \underbrace{\int_{10}^{a} x^{-3}\,dx}_{\text{finite, for fixed } a} = \lim_{a\to\infty}\left(\frac{-1}{2}x^{-2}\Big|_{10}^{a}\right)$$

$$= \lim_{a\to\infty}\left(\frac{-1}{2a^2} - \frac{-1}{2\cdot 10^2}\right) = \frac{1}{200} - \lim_{a\to\infty}\left(\frac{1}{2a^2}\right) = \boxed{\frac{1}{200}}$$

This limit exists, so we call this improper integral <u>convergent</u>.



ex $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$

two sources of impropriety: use 2 limits

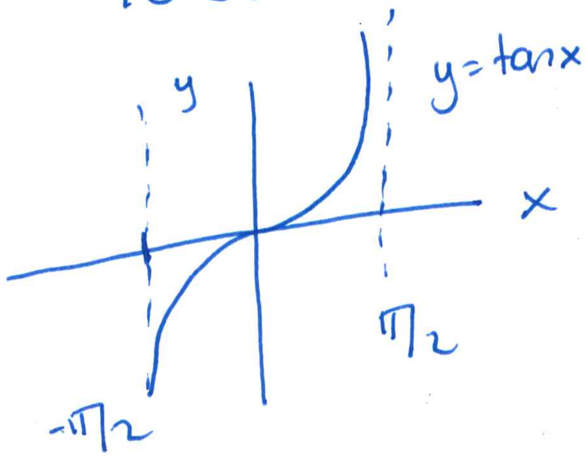
$$= \int_{-\infty}^{3} \frac{1}{1+x^2}\,dx + \int_{3}^{\infty} \frac{1}{1+x^2}\,dx$$

split it up anywhere

$$= \lim_{a\to-\infty} \underbrace{\int_{a}^{3} \frac{1}{1+x^2}\,dx}_{\text{finite for fixed } a} + \lim_{b\to\infty} \underbrace{\int_{3}^{b} \frac{1}{1+x^2}\,dx}_{\text{finite for fixed } b}$$

$$= \lim_{a\to-\infty} \left(\arctan 3 - \arctan a \right) + \lim_{b\to\infty} \left(\arctan b - \arctan 3 \right)$$

To evaluate limits, consider tan / arctan



$y = \tan x \iff x = \arctan y$

If $\tan x \to \to \to -\infty$

then $x \to \to \to -\pi/2$

$$\lim_{x \to -\infty} \underbrace{\arctan x} = -\pi/2$$

" which angle has tangent x ? "

Similarly: $\lim_{x \to \infty} \underbrace{\arctan x}_{\theta} = \pi/2$

As angle gets close to $\pi/2$ (from left), $\tan(\theta) \to \infty$

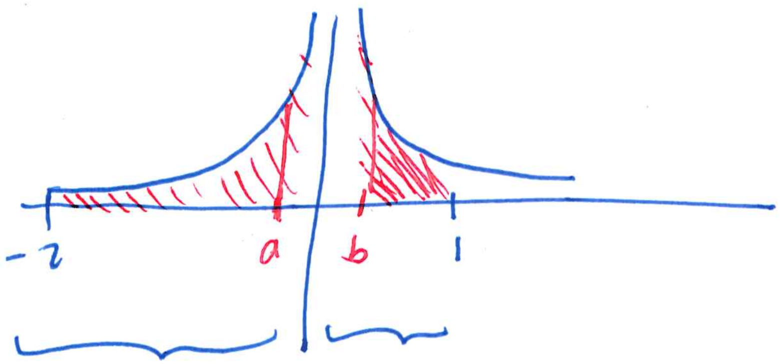
$$\lim_{a \to -\infty} (\arctan 3 - \arctan a) + \lim_{b \to \infty} (\arctan b - \arctan 3)$$

$$= \cancel{\arctan 3} - (-\pi/2) + (\pi/2) - \cancel{\arctan 3}$$

$$= \pi$$

ex $\displaystyle\int_{-2}^{1} \frac{1}{x^2}\,dx$

FTC(II) only applies if integrand continuous



$$= \underbrace{\lim_{a\to 0^-}\left(\int_{-2}^{a} \frac{1}{x^2}\,dx\right)}_{} + \lim_{b\to 0^+}\left(\int_{b}^{1} \frac{1}{x^2}\,dx\right)$$

If one side diverges, we say the whole integral diverges

Left part:

$$\lim_{a\to 0^-}\left[\frac{-1}{x}\Big|_{-2}^{a}\right]$$

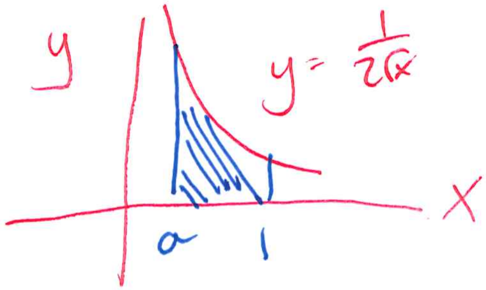
$$= \lim_{a\to 0^-}\left[\underbrace{\frac{-1}{a}}_{\to\infty} - \frac{-1}{-2}\right] \cdot \underline{\underline{\text{DNE}}}$$

b/c one limit DNE, integral <u>diverges</u>

$$\lim_{a\to 0^-} \underbrace{\frac{-1}{a}}_{} = \infty$$

As denom $\to 0$
Fraction blows up

ex $\displaystyle\int_0^1 \frac{1}{2\sqrt{x}}\,dx = \lim_{a\to 0^+} \underbrace{\int_a^1 \frac{1}{2\sqrt{x}}\,dx}_{\text{finite for fixed } a}$



$$= \lim_{a\to 0^+} \left(\sqrt{x} \Big|_a^1 \right)$$

$$= \lim_{a\to 0^+} \left(1 - \underbrace{\sqrt{a}}_{\to 0} \right) = 1 - 0 = \boxed{1}$$

Integral converges

FACTS:

$\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ CONV

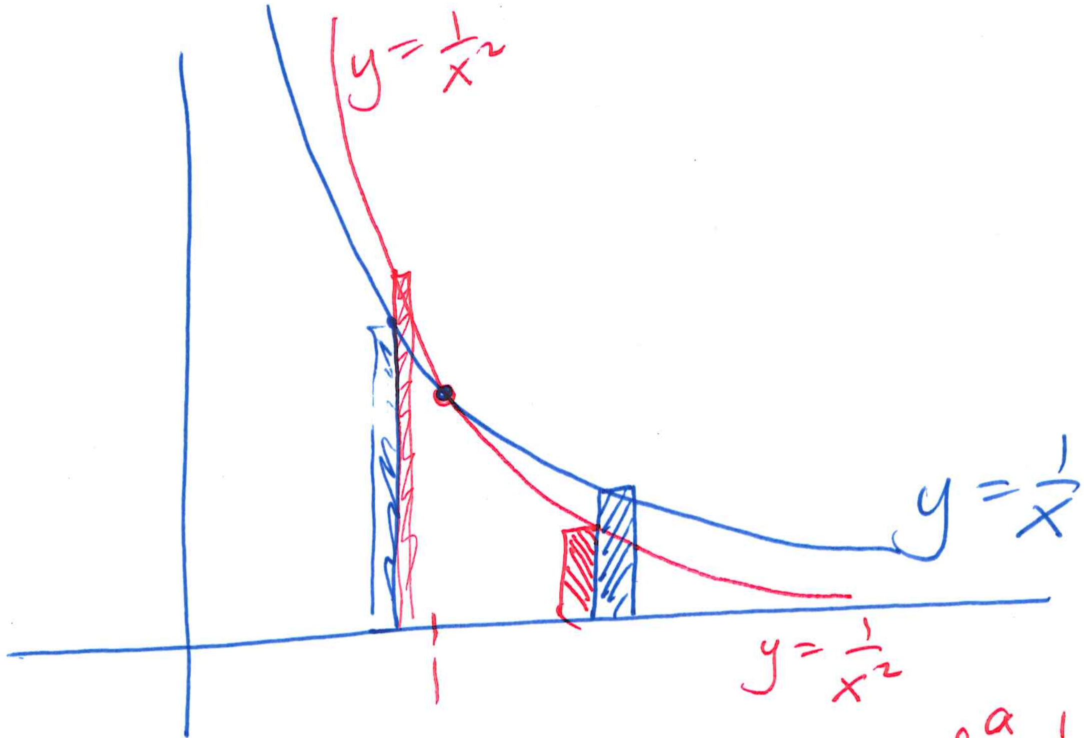
$\displaystyle\int_1^\infty \frac{1}{x}\,dx$ DIV

$\displaystyle\int_1^\infty \frac{1}{x^3}\,dx$ CONV

$\displaystyle\int_0^1 \frac{1}{2x^{1/2}}\,dx$ CONV

$\displaystyle\int_0^1 \frac{1}{x}\,dx$ DIV

$\displaystyle\int_0^1 \frac{1}{x^2}\,dx$ div



If $a > 1$: $\int_1^a \frac{1}{x}\,dx > \int_1^a \frac{1}{x^2}\,dx$

If $a < 1$:

$\int_a^1 \frac{1}{x^2}\,dx > \int_a^1 \frac{1}{x}\,dx$

P-Test

"power"

Let p be a real number.

$$\int_0^1 \frac{1}{x^p}\,dx \qquad \begin{array}{l} \text{conv if } p < 1 \\ \text{div if } p \geq 1 \end{array}$$

$$\int_1^{\text{inf}} \frac{1}{x^p}\,dx \qquad \begin{array}{l} \text{conv if } p > 1 \\ \text{div if } p \leq 1 \end{array}$$

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eg $\displaystyle\int_0^1 \frac{1}{x^{0.001}}\,dx$ $\quad p = 0.001 < 1$

So int converges by p-test

eg $\displaystyle\int_1^\infty \frac{1}{x^{0.999}}\,dx$ $\quad p = 0.999 < 1$

So int diverges by p-test

(ex) $\displaystyle\int_e^\infty \frac{1}{x\ln^2 x}\,dx$

let $u = \ln x$

$du = \frac{1}{x}\,dx$

$= \displaystyle\lim_{a\to\infty} \int_e^a \frac{1}{x\ln^2 x}\,dx$

If $x = e$
$u = \ln x = \ln e = 1$

If $x = a$
$u = \ln x = \ln a$

$= \displaystyle\lim_{a\to\infty} \int_1^{\ln a} \frac{1}{u^2}\,du$

Will $\underline{\text{conv}}$ by p-test
$(p = 2 > 1)$

$= \displaystyle\lim_{a\to\infty} \left[\frac{-1}{u} \Big|_1^{\ln a} \right]$

$= \displaystyle\lim_{a\to\infty} \left[\frac{-1}{\ln a} - \frac{-1}{1} \right] = \lim_{a\to\infty} \left[\frac{-1}{\ln a} + 1 \right] = \left(\lim_{a\to\infty} \frac{-1}{\ln a} \right) + 1$

$= 1 - \displaystyle\lim_{a\to\infty} \left(\frac{1}{\ln a} \right)$ $\to 0$

$\ln a \to \infty$

$= 1$