Remember from Last Class

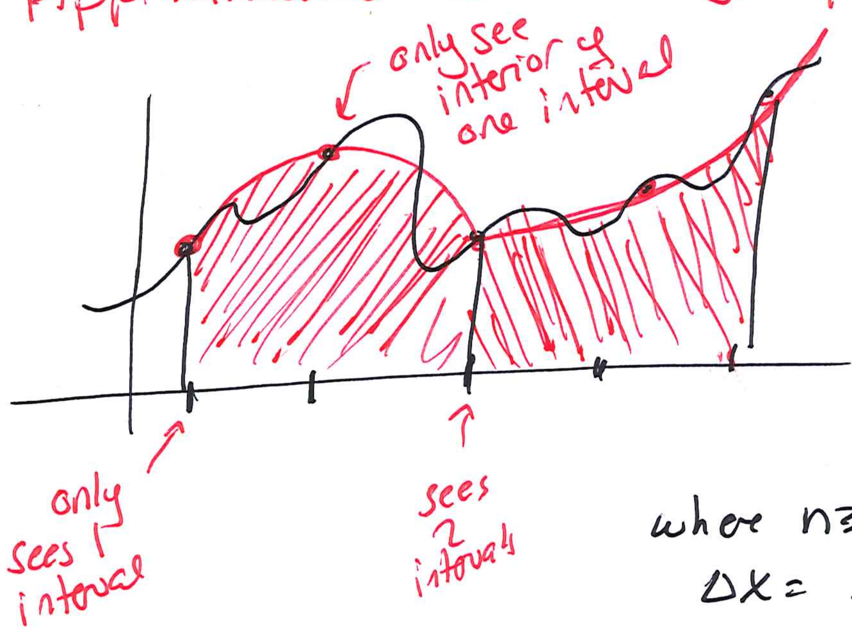
Numerical Integration: approximating a definite integral

Midpoint Rule: $\int_a^b f(x)\,dx \approx \Delta x\left(f(m_1) + f(m_2) + \cdots + f(m_n)\right)$, where $\Delta x = \frac{b-a}{n}$ and m_i is the midpoint of the i^{th} interval

Trapezoid Rule: $\int_a^b f(x)\,dx \approx \Delta x\left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]$

(New) Simpson's Rule

Approximates area using parabolas



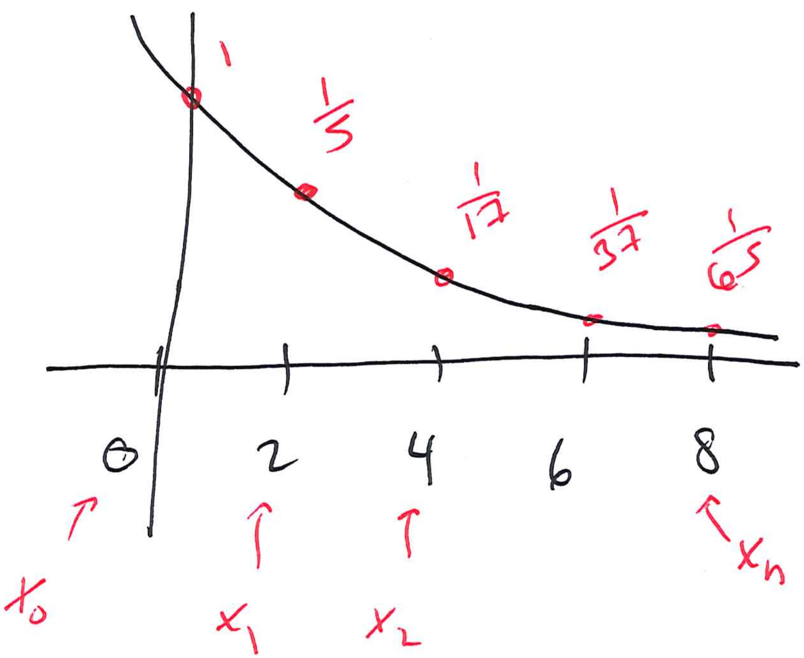
$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\Big[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\Big]$$

where $n \geq 2$ even

$\Delta x = \frac{b-a}{n}$, $x_k = a + k\Delta x$

(ex) Approx $\int_0^8 \frac{1}{1+x^2}\,dx$ using $n=4$, Simpson's Rule:

$\Delta x = \frac{8-0}{4} = 2$ width



Area $\approx$

$$\frac{2}{3}\left[1+4\left(\frac{1}{5}\right)+2\left(\frac{1}{17}\right)+4\left(\frac{1}{37}\right)+\frac{1}{65}\right]$$

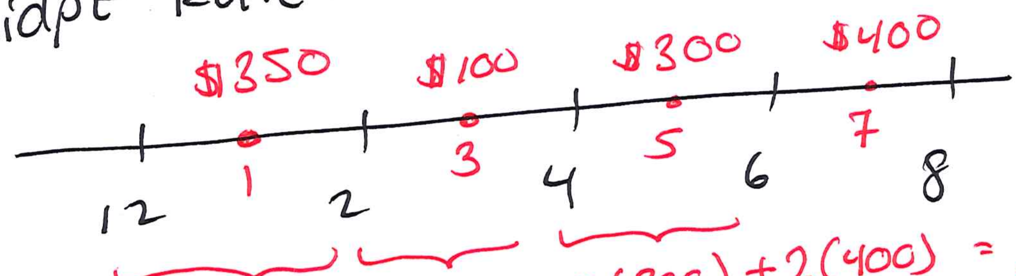
Coeff: 1 4 2 4 ... 1

(ex) Your labour costs fluctuate throughout the day. Instaneous rates per hour given:

time	12	1	2	3	4	5	6	7	8
rates	\$300	\$350	\$150	\$100	\$100	\$300	\$400	\$400	\$200

From 12 to 8, how much spent on labour? (Use $n=4$)

Midpt Rule:



$$2(350)+2(100)+2(300)+2(400) = \underbrace{\Delta x}_{2}\left(f(1)+f(3)+f(5)+f(7)\right)$$

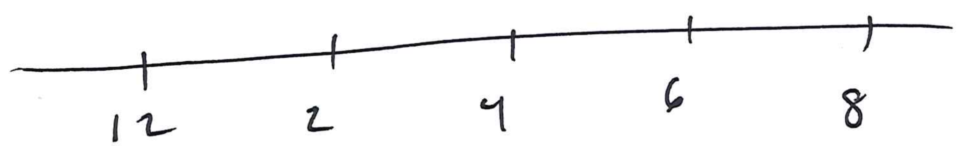
$$= 700+200+600+800 = \$2300$$ approx daily labour cost

Trapezoid Rule: $\int_a^b f(x)\,dx \approx \Delta x\left(\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}f(x_4)\right)$

Δx width of intervals, 2

(8 hrs, cut into 4 pieces)

$$\text{cost} \approx 2\left(\tfrac{1}{2}f(12) + f(2) + f(4) + f(6) + \tfrac{1}{2}f(8)\right)$$



$$= 2\left(\underbrace{\tfrac{1}{2}\cdot 300 + 150}_{} + \underbrace{100 + 400}_{} + \tfrac{1}{2}200\right)$$

$$= 2(300 + 500 + 100) = \$1800$$ for the day

Simpson's Rule:

$$\frac{2}{3}\left(1f(12) + 4f(2) + 2f(4) + 4f(6) + 1f(8)\right)$$

$2 = \Delta x$

$$= \frac{2}{3}\left(300 + 4\cdot 150 + \underbrace{2\cdot 100 + 4\cdot 400 + 200}\right)$$

$$= \frac{2}{3}\left(300 + 600 + 2000\right)$$

$$= \frac{2}{3}(2900) = \frac{2}{3}(3000 - 100) = 2000 - \frac{2}{3}(100)$$

$$\approx 2000 - 67 = \$1933$$ for the day

We need to be able to bound our error

ex) If we approximate $\int_0^1 \sin(2x)\,dx$ using midpt rule, $n=10$. How bad could our error be?

$\boxed{\text{PAGE 565}}$ Theorem (will be given if you need it)

$f(x) = \sin(2x)$

Find k such that $|f''(x)| \le k$ for all x in $[0,1]$

$f'(x) = 2\cos(2x)$

$f''(x) = -4\sin(2x)$

$|f''(x)| = |-4\sin(2x)|$

$= 4\,|\sin(2x)| \le 4 \cdot 1 = 4$

Use $k=4$

Thm: $E_M \le \dfrac{k(b-a)^3}{24n^2}$

absolute error MP rule

$= \dfrac{4(1-0)^3}{24 \cdot 10^2} = \dfrac{4}{24 \cdot 10^2} = \dfrac{1}{600}$

ex If we approximate $\int_2^3 \frac{1}{x}\,dx$ using $n=6$, Simpson's rule, what is worst possibe error?

$n=6$
$b-a = 3-2$

Thm, p565: $E_S \leq \dfrac{K(b-a)^5}{180n^4}$

absolute error, Simpson

where $|f^{(4)}(x)| \leq K$

find a suitable K

① Find 4^{th} deriv of $\frac{1}{x}$

② If $2 \leq x \leq 3$, how big can 4^{th} deriv be?

$f(x) = x^{-1}$
$f'(x) = -x^{-2}$
$f''(x) = 2x^{-3}$
$f'''(x) = -6x^{-4}$
$f^{(4)}(x) = 24x^{-5}$

So, if $2 \leq x \leq 3$,

$$|f^{(4)}(x)| = \left|\frac{24}{x^5}\right| \leq \frac{24}{2^5} = \frac{8 \cdot 3}{8 \cdot 4} = \boxed{\frac{3}{4} = K}$$

x in denom:

$\frac{24}{x^5}$ gets big as x gets small

(when $x=2$)

③ Plug in!

$$E_S \le \frac{K(b-a)^5}{180n^4}$$

$$= \frac{3/4\,(3-2)^5}{180 \cdot 6^4}$$

$$= \frac{3}{4 \cdot 6^4 \cdot 180} \sim 0.0007 \sim 0.000003$$

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ex) To approx $\int_1^2 \frac{1}{x}\,dx$ to within an error of 10^{-4} ← tolerance (your needs — machinery, etc)

using midpt rule, what should n be?

Theorem: $E_M \le \boxed{\frac{k(b-a)^3}{24n^2} \le 10^{-4}}$ Solve for n

want

Find k such that $|f''(x)| \le k$ for $1 \le x \le 2$

$f(x) = x^{-1}$

$f'(x) = -x^{-2}$

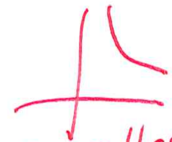
$f''(x) = 2x^{-3}$

What is max of $2x^{-3}$ over $[1, 2]$

(optimization)

$$\left|\frac{2}{x^3}\right| \leq \frac{2}{1^3} = \boxed{2 = k}$$

Decr fcn



Max @ smallest x

$$\frac{k(b-a)^3}{24n^2} < \frac{1}{10^4}$$

$$\frac{2\cancel{(2-1)^3}}{24n^2} < \frac{1}{10^4}$$

$n^2 10^4$

$$\frac{2}{24} \cdot 10^4 < n^2$$

$\sqrt{}$

$$\frac{1}{\sqrt{12}} \cdot 10^2 < n$$

exam answer form

Note $\frac{10^2}{\sqrt{12}} \approx 28.8$

need $n > 28.8$

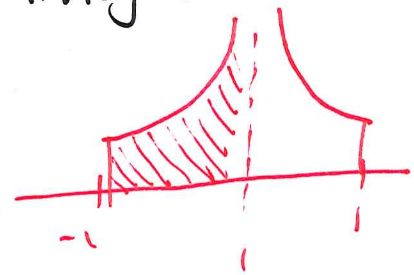
(need n : integer)

$n = 29$ (or higher works)

Ch 7.8: Improper Integrals

2 ways for an integral to be improper:

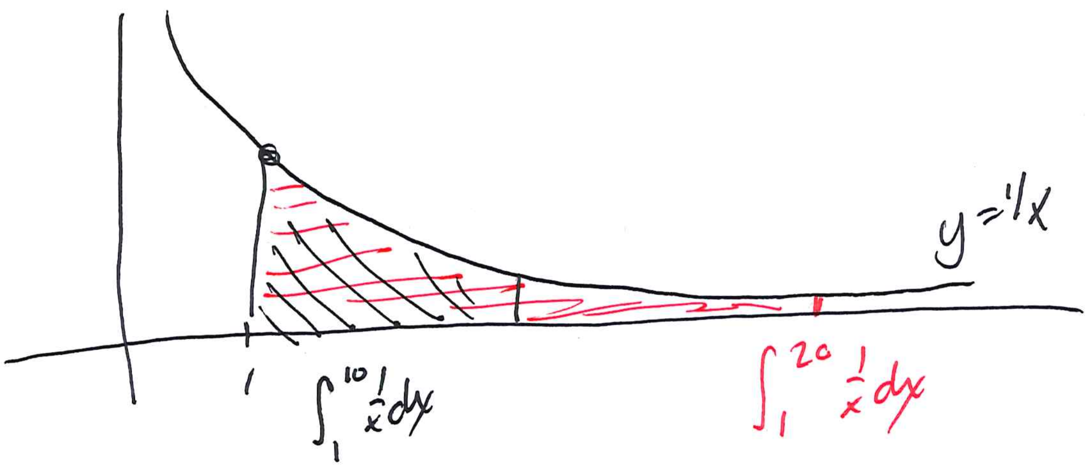
- unbounded region of integration eg $\int_1^{\infty} \frac{1}{x}\,dx$
- unbounded integrand over region of integration eg $\int_{-1}^{1} \frac{1}{x}\,dx$



Idea: $\int_1^{\infty} \frac{1}{x}\,dx = \lim_{a\to\infty} \int_1^{a} \frac{1}{x}\,dx = \lim_{a\to\infty} \left[\ln x \Big|_1^a \right]$

$= \lim_{a\to\infty} \left[\ln a - \ln 1 \right]$

$= \lim_{a\to\infty} \ln a = \infty$



We say this integral <u>diverges</u> b/c limit DNE

(∞ not a number)

ex

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{a\to\infty}\int_1^a \frac{1}{x^2}\,dx = \lim_{a\to\infty}\left[\frac{-1}{x}\Big|_1^a\right]$$

$$= \lim_{a\to\infty}\left[\frac{-1}{a} - \frac{-1}{1}\right]$$

$$= \lim_{a\to\infty}\left[1-\underbrace{\frac{1}{a}}_{\to 0}\right] = 1$$