# Remember from Last Time:

Midpoint Rule: $\int_a^b f(x)dx \approx \Delta x f(m_1) + \Delta x f(m_2) + \cdots + \Delta x f(m_n)$

where $\Delta x = \frac{b-a}{n}$, and $m_i = a + (i - 1/2)\Delta x$ is the midpt of the $i^{th}$ interval

Trapezoid Rule: $\int_a^b f(x)dx \approx \Delta x \left[\frac{1}{2} f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2} f(x_n)\right]$

where $x_i = a + i\Delta x$

Simpson's Rule: $\int_a^b f(x)dx \approx \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 4f(x_{n-1}) + f(x_n)\right]$

for an even $n$

Note: We've got a lot of material to cover before the end of term! If you have a hard time keeping up in lecture, it might help you to skim the text before class. That will help you to focus on the parts you don't understand yet.

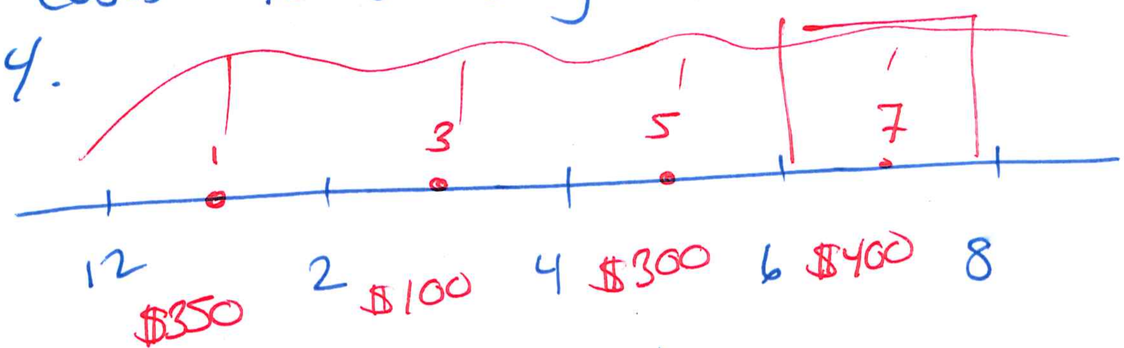
ex Your labour costs vary throughout the day. Instantaneous rate per hour:

| time | 12 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| rate | \$300 | \$350 | \$150 | \$100 | \$100 | \$300 | \$400 | \$400 | \$200 |

Approx labour costs 12–8 using all three methods

$n=4$.

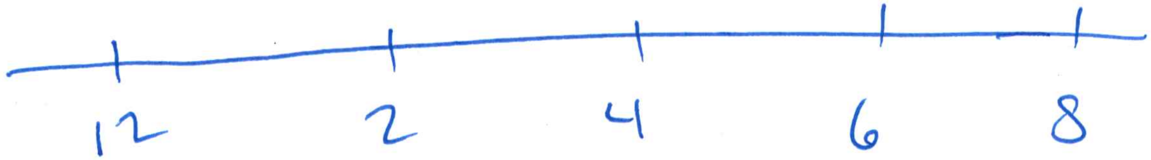
MIDPT RULE:




$$\text{Cost} \approx 2(350)+2(100)+2(300)+2(400)$$

$$700+200+600+800 = \$2300$$

Trapezoid Rule:




$$2\left(\frac{1}{2}f(12)+f(2)+f(4)+f(6)+\frac{1}{2}f(8)\right)$$

$$=2\left(\frac{1}{2}\cdot 300+150+100+400+\frac{1}{2}(200)\right)$$

$$=2(150+150+600)=2(900)=\$1800$$

Simpson's Rule $\frac{2}{3}\left( f(12) + 4f(2) + 2f(4) + 4f(6) + f(8) \right)$

$1 \quad 4 \quad 2 \quad 4 \quad 1$

$= \frac{2}{3}\left( 300 + 4 \cdot 150 + 2 \cdot 100 + 4 \cdot 400 + 200 \right)$

$= \frac{2}{3}\left( 300 + 600 + 1800 + 200 \right)$

$= \frac{2}{3}(2900) < \frac{2}{3}(3000) = 2000$

# We need to talk about error

Absolute error: $|\text{exact} - \text{approx}|$

Relative error: $\dfrac{\text{Abs error}}{|\text{exact}|}$

ex Actual: 500, Approx: 495

Abs error: $|500 - 495| = 5$

Rel error: $\frac{5}{500} = \frac{1}{100} = 1\%$

ex Actual: 5 Approx: 10

Abs error: 5

Rel error: $\frac{5}{5} = 1 = 100\%$

same absolute error

diff rel error

(ex) We plan to approximate $\int_0^1 \sin(2x)\,dx$ using $n=10$ and Midpt Rule.

What will my error be?

PAGE 565 (will be given to you)

$|f''(x)| \le k$ on $[0,1]$

Find such a $k$:

$f(x) = \sin 2x$

$f'(x) = 2\cos(2x)$

$f''(x) = -4\sin(2x)$

$|-4\sin 2x| \le 4$

because $|\sin(\ )| \le 1$

We take $k=4$

Theorem:

$$\text{Error} \le \frac{k(b-a)^3}{24n^2} = \frac{4(1-0)^3}{24 \cdot 10^2} = \frac{4}{24 \cdot 100} = \boxed{\frac{1}{600}}$$

Using $n=10$, MP rule to approx $\int_0^1 \sin(2x)\,dx$

I'll have error no worse than $\frac{1}{600}$

ex Approx $\int_2^3 \frac{1}{x}\,dx$ using Simpson's Rule, $n=6$

What is error bound?

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$$E_S \le \frac{K(b-a)^5}{180n^4}$$

where: $|f^{(4)}(x)| \le K$ for $2 \le x \le 3$

Find a suitable $K$

$f(x) = \frac{1}{x} = x^{-1}$

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$

$f'''(x) = -6x^{-4}$

$f^{(4)}(x) = 24x^{-5}$

$$\frac{24}{x^5} \le \frac{24}{2^5} = \boxed{\frac{3}{4} =: K}$$

As $x$ increases $\frac{24}{x^5}$ decreases

Max of $\frac{24}{x^5}$ occurs at min value of $x$

Theorem: $E_S \leq \dfrac{K(b-a)^5}{180 n^4} = \dfrac{3/4(3-2)^5}{180 \cdot 6^4}$

error

$= \dfrac{3}{4 \cdot 180 \cdot 6^4} \approx 0.0000003$

worst-case scenario, this is my error

ex You want to approximate $\int_1^2 \frac{1}{x}\,dx$ using the midpt rule with error $\leq 10^{-4}$. How many intervals do you need?

$|f''(x)| \leq k$

Theorem: $E_m \leq \underbrace{\dfrac{k(b-a)^3}{24 n^2} \leq 10^{-4}}$

want

find k then solve for n

$f(x) = \frac{1}{x}$ If $1 \le x \le 2$, then

$f'(x) = -x^{-2}$ $\left|\frac{2}{x^3}\right| \le 2$ Let $k=2$

$f''(x) = 2x^{-3}$

smaller $x \to$ larger $\frac{2}{x^3}$

$$\frac{2(2-1)^3}{24n^2} \le 10^{-4}$$

$$\frac{2}{24n^2} \le \frac{1}{10^4} \Rightarrow \frac{1}{12n^2} \le \frac{1}{10^4}$$

$$\Rightarrow 10^4 \le 12n^2$$

$$\Rightarrow \frac{10^4}{12} \le n^2$$

$$n \ge \frac{10^2}{\sqrt{12}} \approx 28.8$$

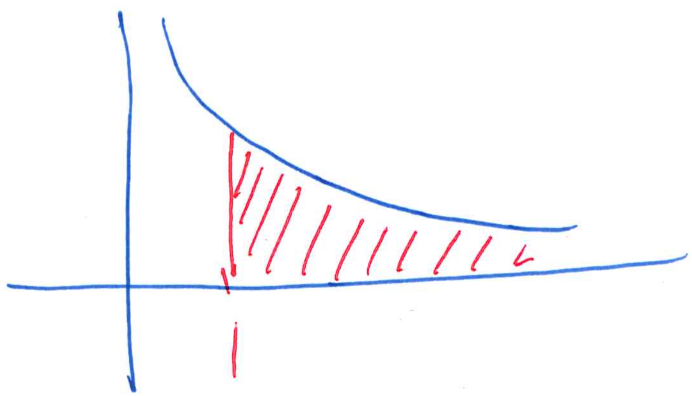
(exam ans)

Use $n=29$ or higher

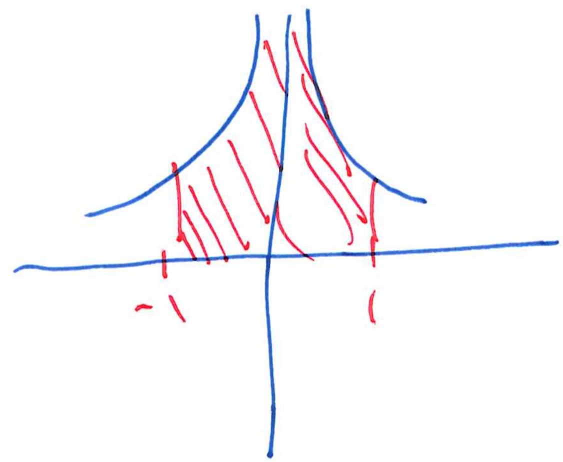
# Ch 7.8: Improper Integrals

There are two ways for an integral to be improper: infinite bound of integration; integrand unbounded over region of integration

eg $\int_1^\infty \frac{1}{x}\,dx$




eg $\int_{-1}^{1} \frac{1}{x^2}\,dx$

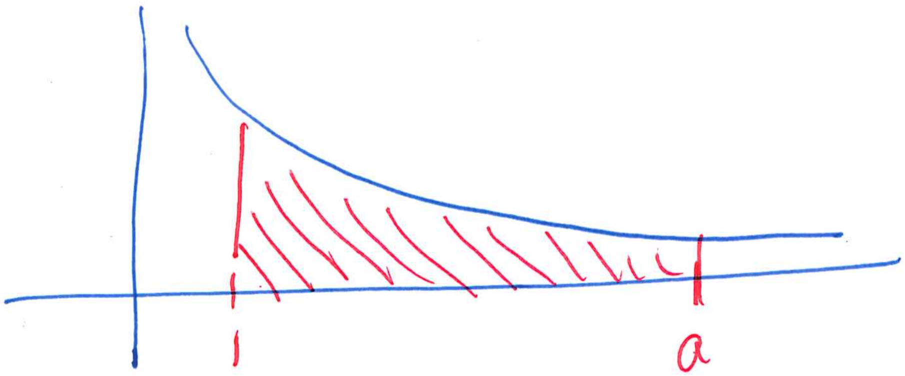



We use a limit to make sense of these.

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{a\to\infty} \int_1^a \frac{1}{x}\,dx = \lim_{a\to\infty} \left[ \ln|x| \Big|_1^a \right]$$

$$= \lim_{a\to\infty} (\ln a - \ln 1)$$

$$= \lim_{a\to\infty} \ln a = \infty$$




The integral <u>diverges</u>

(b/c limit DNE)

ex $$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{a\to\infty} \int_1^a x^{-2}\,dx = \lim_{a\to\infty} \left[ -x^{-1} \Big|_1^a \right]$$

$$= \lim_{a\to\infty} \left[ \frac{-1}{a} - \frac{-1}{1} \right] = \lim_{a\to\infty} \left[ 1 - \underset{\to 0}{\frac{1}{a}} \right] = 1$$

We say this <u>converges</u>

(limit real #)