Method of Partial Fractions

For antidifferentiating $\underbrace{\text{rational functions}}_{\frac{\text{polynomial}}{\text{polynomial}}}$ when a substitution is not available.

We re-write our rational function as a sum of other rational functions that are more easily integrable

ex $\dfrac{x-2}{(x+1)(2x-1)} = \dfrac{1}{x+1} - \dfrac{1}{2x-1}$, so we can replace

$\displaystyle\int \frac{x-2}{(x+1)(2x-1)}\,dx$ with $\displaystyle\int \left(\frac{1}{x+1} - \frac{1}{2x-1}\right) dx$, which is easier to evaluate.

Fine Print: some examples in today's lecture may be easier to evaluate using other methods. We use them just to learn the method of partial fractions.

CASE 1: DENOM: REPEATED LINEAR FACTORS

$$\frac{\text{num}}{(ax+r)^n} = \frac{A_1}{ax+r} + \frac{A_2}{(ax+r)^2} + \frac{A_3}{(ax+r)^3} + \cdots + \frac{A_n}{(ax+r)^n}$$

num: polynomial, degree $< n$

constant A, B, C

ex $$\frac{x^2}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$ Find A, B, C

repeated linear factor in denom

deg of top < deg of bottom
(2) (3)

$$= \frac{A(x-1)^2}{(x-1)^3} + \frac{B(x-1)}{(x-1)^3} + \frac{C}{(x-1)^3}$$

Add right-hand side (common denom)

$$= \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

So: $x^2 = A(x-1)^2 + B(x-1) + C$

Equal polynomials have matching coefficients of powers of x

eg $x^2 + 1 = ax^2 + bx + C$

$a = 1$, $b = 0$, $c = 1$

So: $x^2 + 0x + 0 = A(x^2 - 2x + 1) + Bx - B + C$

group like terms

$$= Ax^2 - 2Ax + A + Bx - B + C$$

$$= (A)x^2 + (-2A + B)x + (A - B + C)$$

$1 = A$

$0 = -2A + B$ $0 = -2 + B$, so $B = 2$

$0 = A - B + C \rightarrow$ $0 = 1 - 2 + C$, so $C = 1$

So: $$\boxed{\frac{x^2}{(x-1)^3} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{(x-1)^3}}$$

new found fact!

Use: $$\int \frac{x^2}{(x-1)^3}\,dx = \int \left[\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{(x-1)^3}\right] dx$$

$u = x - 1$
$du = dx$

$$= \int \left(\frac{1}{u} + 2u^{-2} + u^{-3}\right) du = \ln|u| - 2u^{-1} - \frac{1}{2}u^{-2} + C$$

$$= \boxed{\ln|x+1| - \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C}$$

CASE 2: DISTINCT LINEAR FACTORS IN DENOMINATOR

$$\frac{\text{num}}{(a_1x-r_1)(a_2x-r_2)\cdots(a_nx-r_n)} = \frac{A_1}{a_1x-r_1} + \frac{A_2}{a_2x-r_1} + \cdots + \frac{A_n}{a_nx-r_n}$$

num: polynomial, deg $< n$ $\quad A_1, \dots, A_n$: constants

(ex) $$\frac{7x+13}{(2x+5)(x-2)} = \frac{A}{2x+5} + \frac{B}{x-2}$$ Find A, B

Denom has distinct linear factors

$$= \frac{A(x-2)}{(2x+5)(x-2)} + \frac{B(2x+5)}{(2x+5)(x-2)}$$ Common Denom

$$= \frac{A(x-2)+B(2x+5)}{(2x+5)(x-2)}$$

So: $7x+13 = A(x-2) + B(2x+5)$ Find A, B

Method 1: group like terms

$$7x + 13 = Ax - 2A + 2Bx + 5B$$

$$= (A+2B)x + (-2A+5B)$$

$7 = A + 2B \longrightarrow A = 7 - 2B$

$13 = -2A + 5B \longrightarrow 13 = -2\underbrace{(7-2B)}_{A} + 5B$

$13 = -14 + 4B + 5B$

$27 = 9B$

$B = 3$

$A = 7 - 2(3)$

$A = 1$

Method 2: choose convenient x-values

works well w/ distinct factors, not so well with repeated factors.

$$7x + 13 = A(x-2) + B(2x+5)$$

when $x = 2$:

roots of denom

$14 + 13 = A(0) + B(9)$

$27 = 9B$

$B = 3$

when $x = -5/2$:

$7\left(\frac{-5}{2}\right) + 13 = A\left(\frac{-5}{2} - 2\right) + B(0)$

$\frac{-35+26}{2} = A\left(\frac{-5-4}{2}\right)$

$-9 = A(-9)$ so $A = 1$

So: $$\frac{7x+13}{(2x+5)(x-2)} = \frac{1}{2x+5} + \frac{3}{x-2}$$

$u = 2x+5$ $\leftarrow w = x-2$

Use: $$\int \frac{7x+13}{(2x+5)(x-2)}\,dx = \int \frac{1}{2x+5}\,dx + \int \frac{3}{x-2}\,dx$$

$$= \frac{1}{2}\ln|2x+5| + 3\ln|x-2| + C$$

CASE 3: MIX CASES 1 & 2

ex $$\frac{x^2+5}{(x-1)^3(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x-2}$$

ex $$\frac{x^3-2x+1}{(x-2)(x-1)^2x^2} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{(x-1)^2} + \frac{D}{x} + \frac{E}{x^2}$$

ex $\dfrac{x^2+4x-7}{(x-2)(x-1)}$ deg of num = deg of denom

First: Long division

$$\frac{x^2+4x-7}{x^2-3x+2} = \frac{(x^2-\overbrace{3x+2) + 7x}^{4x} \underbrace{- 9}_{-7}}{x^2-3x+2} = \frac{x^2-3x+2}{x^2-3x+2} + \frac{7x-9}{x^2-3x+2}$$

$$= 1 + \frac{7x-9}{(x-2)(x-1)}$$

use partial fractions on this part

Done w/ integration methods!

sub, parts, trig sub, partial fractions, ...

Ch 7.7 : Numerical Integration

Approximating definite integrals in a way that controls the error.

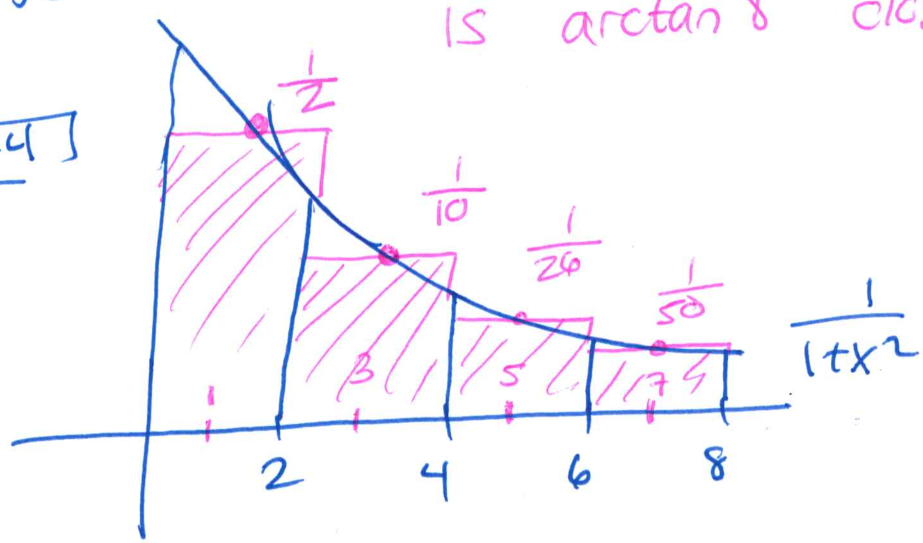
3 Methods: Midpt Rule, Trapezoid Rule, Simpson's Rule.

Midpt Rule: like midpt Riemann Sum

ex $\int_0^8 \frac{1}{1+x^2}\,dx = \arctan 8 - \arctan 0 = \arctan 8$

Is arctan 8 close to 1? 1.5? 1.25?

$\boxed{n=4}$



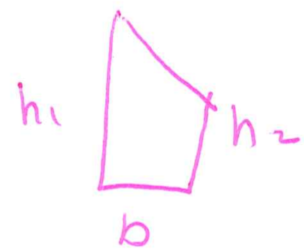
BASE: 2

$$\arctan 8 = \int_0^8 \frac{1}{1+x^2}\,dx = 2\cdot\frac{1}{2} + 2\cdot\frac{1}{10} + 2\,\frac{1}{26} + 2\,\frac{1}{50} \approx 1.3$$

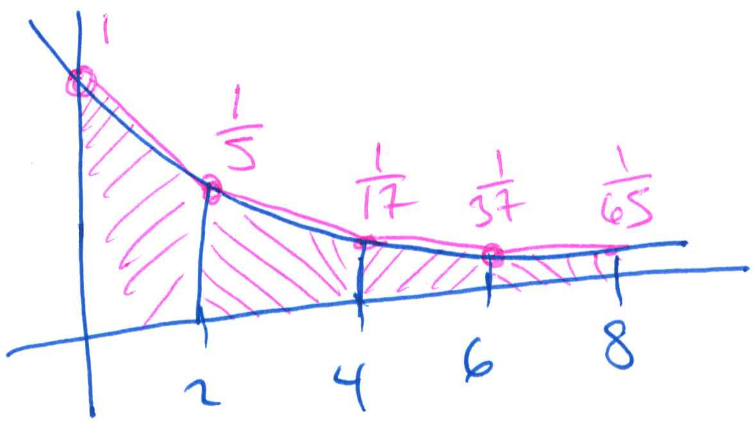
(base)(height)

Trapezoid Rule: Trapezoids, not rectangles

$$\int_0^8 \frac{1}{1+x^2}\,dx$$



Area: $\frac{1}{2}b(h_1+h_2)$



$$A \approx \underbrace{\frac{1}{2}(2)\left(1+\frac{1}{5}\right)} + \underbrace{\frac{1}{2}(2)\left(\frac{1}{5}+\frac{1}{17}\right)}$$

$$+\ \frac{1}{2}(2)\left(\frac{1}{17}+\frac{1}{37}\right) + \frac{1}{2}2\left(\frac{1}{37}+\frac{1}{65}\right) \approx 1.6$$

Trapezoid Rule:

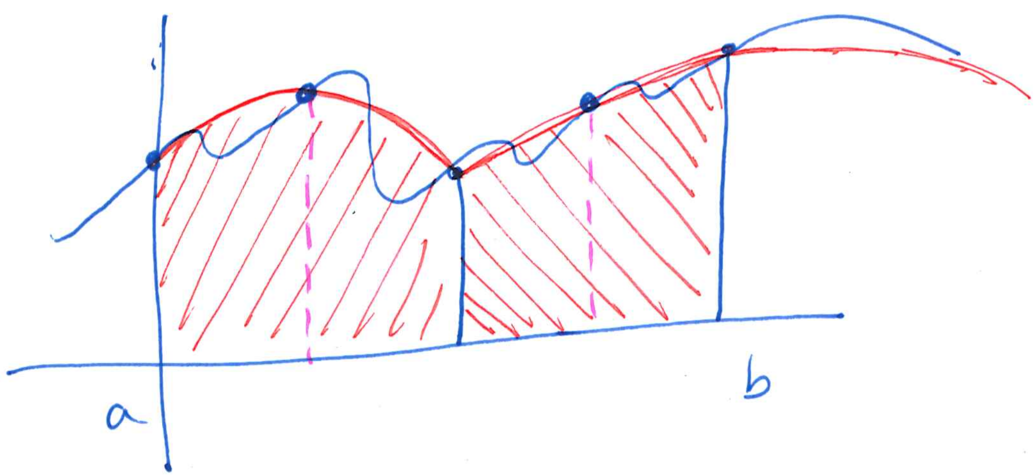
$$\int_a^b f(x)\,dx \approx \frac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)\right)$$

$\Delta x = \frac{b-a}{n}$ "base"

$$= \sum_{i=1}^{n-1} \Delta x\, f(x_i) + \frac{\Delta x}{2} f(x_0) + \frac{\Delta x}{2} f(x_n)$$

Simpson's Rule: approximates using parabolas (instead of lines)

n: even, $n \geq 2$



$$\int_a^b f(x)\,dx \approx$$

$$\frac{\Delta x}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_3) \dots + 4f(x_{n-1}) + f(x_n)\Big)$$

$$\int_0^8 \frac{1}{1+x^2}\,dx \approx \frac{2}{3}\left(f(0)+4f(2)+2f(4)+4f(6)+f(8)\right)$$

$$= \frac{2}{3}\left(1+\frac{4}{5}+\frac{2}{17}+\frac{4}{37}+\frac{1}{65}\right) \approx 1.4$$

$n=4$