Ch 7.5 : Integration by Method of Partial Fractions

ex $\displaystyle\int \frac{x-2}{(x+1)(2x-1)}\,dx =$

Useful fact : $\underbrace{\frac{x-2}{(x+1)(2x-1)} = \frac{1}{x+1} - \frac{1}{2x-1}}_{\text{algebra (today)}}$

$\underbrace{\displaystyle\int \frac{1}{x+1}\,dx}_{\substack{u=x+1\\ du=dx}} - \underbrace{\displaystyle\int \frac{1}{2x-1}\,dx}_{\substack{u=2x-1 \to du=2dx\\ \frac{1}{2}du=dx}}$

$\displaystyle\int \frac{1}{u}\,du - \int \frac{1}{2}\cdot\frac{1}{u}\,du$

$\ln|u| - \frac{1}{2}\ln|u| + C$

$\boxed{\ln|x+1| - \frac{1}{2}\ln|2x-1| + C}$

$\frac{1}{x+1} - \frac{1}{2x-1} =$

$\frac{(2x-1)}{(x+1)(2x-1)} - \frac{(x+1)}{(x+1)(2x-1)} =$

$\frac{2x-1-x-1}{(x+1)(2x-1)} = \frac{x-2}{(x+1)(2x-1)}$

Method of Partial Fractions:

re-write a $\underbrace{\text{rational function}}_{\frac{\text{polynomial}}{\text{polynomial}}}$ as a sum of fractions that is easier to antidifferentiate.

CASE 1: REPEATED LINEAR FACTORS IN DENOMINATOR

$$\frac{\text{num}}{(ax-r)^n} = \frac{A_1}{ax-r} + \frac{A_2}{(ax-r)^2} + \frac{A_3}{(ax-r)^3} + \cdots + \frac{A_n}{(ax-r)^n}$$

deg of num $< n$ $\qquad$ $A_1, \ldots, A_n$ constants

ex $\displaystyle\int \frac{x^2}{(x-1)^3}\,dx$

- rational fcn
- deg of num $= 2$, deg of denom $= 3$
  $2 < 3$
- denom: repeated lin factor $(x-1)$

$$\frac{x^2}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$ } Setup from note above

$$= \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

Now: find A, B, C

So: $x^2 = A(x-1)^2 + B(x-1) + C$

$1x^2 + 0x + 0 = A(x^2 - 2x + 1) + Bx - B + C$

$= Ax^2 + x(-2A+B) + (A-B+C)$

(grouping)

If two polynom's are equal, their coeffs match

eg $ax^2 + bx + c = 27x^2 + 5 + 0x$

Then: $a = 27$
$b = 0$
$5 = c$

$Ax^2 - 2Ax + A + Bx - B + C$

$Ax^2 + Bx - 2Ax + A - B + C$

$Ax^2 + (B-2A)x + (A-B+C)$

MATCH COEFFS:

$1 = A$

$0 = -2A + B$

$0 = A - B + C$

$0 = -2 + B$, so $B = 2$

$0 = 1 - 2 + C$

so $C = 1$

My Newfound Fact:

$$\left[\frac{x^2}{(x-1)^3} = \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{(x-1)^3}\right]$$

Can check by adding RHS

So:

$$\int \frac{x^2}{(x-1)^3}\,dx = \int \left[\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{1}{(x-1)^3}\right] dx$$

$u = x-1$
$du = dx$

$$= \int \left(\frac{1}{u} + 2u^{-2} + u^{-3}\right) du = \ln|u| - 2u^{-1} - \frac{1}{2}u^{-2} + C$$

$$= \boxed{\ln|x-1| - \frac{2}{x-1} - \frac{1/2}{(x-1)^2} + C}$$

(ex) $\displaystyle\int \frac{6x+7}{4x^2+20x+25}\,dx$

$\underbrace{\hphantom{4x^2+20x+25}}$

Rational fcn

deg of num < deg of denom

denom: $(2x+5)^2$: repeated linear factor

Method says:

$$\frac{6x+7}{(2x+5)^2} = \frac{A}{2x+5} + \frac{B}{(2x+5)^2}$$

Find A, B

$$\frac{6x+7}{\cancel{(2x+5)^2}} = \frac{A}{2x+5}\left(\frac{2x+5}{2x+5}\right) + \frac{B}{(2x+5)^2}$$

$$= \frac{A(2x+5)+B}{(2x+5)^2}$$

$$= \frac{2Ax+5A+B}{\cancel{(2x+5)^2}}$$

$$6x + 7 = 2Ax + 5A + B$$

$6 = 2A$, so $A = 3$

$7 = 5A + B = 15 + B$

$B = 7 - 15 = -8$

New Found Fact:

$$\frac{6x+7}{(2x+5)^2} = \frac{3}{2x+5} - \frac{8}{(2x+5)^2}$$

$$\int \frac{6x+7}{4x^2+20x+25}\,dx = \int \frac{3}{2x+5} - \frac{8}{(2x+5)^2}\,dx$$

$u = 2x+5$
$du = 2dx$
$\frac{1}{2}du = dx$

$$= \int \left(\frac{3}{u} - \frac{8}{u^2}\right)\frac{1}{2}\,du$$

$$= \frac{1}{2}\int \left(\frac{3}{u} - 8u^{-2}\right)du = \frac{1}{2}\left[3\ln|u| + 8u^{-1}\right] + C$$

$$= \boxed{\frac{1}{2}\left[3\ln|2x+5| + \frac{8}{2x+5}\right] + C}$$

CASE 2: DISTINCT LINEAR ROOTS

$$\frac{\text{num}}{(a_1x+r_1)(a_2x+r_2)\cdots(a_nx+r_n)} = \frac{A_1}{a_1x+r_1} + \frac{A_2}{a_2x+r_2} + \cdots + \frac{A_n}{a_nx+r_n}$$

deg num < n

ex

$$\int \frac{7x+13}{2x^2+x-10}\,dx = \int \frac{7x+13}{(2x+5)(x-2)}\,dx$$

Case 2: lin factors different

Method: $$\frac{7x+13}{(2x+5)(x-2)} = \frac{A}{2x+5} + \frac{B}{x-2}$$

$$= \frac{A(x-2)+B(2x+5)}{(2x+5)(x-2)}$$

Find A, B

↳ can find by matching coeffs

↳ can also find using clever values of x

$$7x+13 = A(x-2)+B(2x+5)$$

Polynoms same ⇒

" " for every x

If $x=2$:

$$14+13 = A(0)+B(9)$$

$$27 = 9B \rightarrow B=3$$

$$7x+13 = A(x-2) + 3(2x+5) \qquad \text{If } x = -5/2$$

$$7\left(\frac{-5}{2}\right)+13 = A\left(\frac{-5}{2}-2\right)+0$$

$$\frac{-35}{\cancel{2}} + \frac{26}{\cancel{2}} = A\left(\frac{-5-4}{\cancel{2}}\right)$$

$$-35+26 = A(-9)$$

$$-9 = A(-9) \qquad \text{so } A=1$$

$$\int \frac{7x+13}{2x^2+x-10}\,dx = \int\left(\frac{1}{2x+5}+\frac{3}{x-2}\right)dx = \boxed{\frac{1}{2}\ln|2x+5| + 3\ln|x-2| + C}$$

ex $\int \frac{10x-11}{4x^2-10x+4}\,dx = \int \frac{10x-11}{(2x-1)(2x-4)}\,dx$

$(2x-1)(2x-4) =$

$(2x-1)(2)(x-2) =$

$(4x-2)(x-2)$

Method :

$\frac{10x-11}{(2x-1)(2x-4)} = \frac{A}{2x-1} + \frac{B}{2x-4}$ Find A, B

$= \frac{A(2x-4) + B(2x-1)}{(2x-1)(2x-4)}$

$10x-11 = A(2x-4) + B(2x-1)$

$10x-11 = x(2A+2B) + (-4A-B)$

So : $10 = 2A+2B$

$-11 = -4A-B \rightarrow B = 11-4A$

$10 = 2A + 2(11-4A)$

$10 = 2A + 22 - 8A$

$\cancel{6A = 12}$

$A = 2$

("grouping" way)

$B = 11 - 8 = 3$

So: $\int \frac{10x-11}{4x^2-10x+4}\,dx = \int \left(\frac{2}{2x-1} + \frac{3}{2x-4}\right) dx$

$u = 2x-1$
$du = 2dx$
$\frac{1}{2}du = dx$

$w = 2x-4$
$dw = 2dx$
$\frac{1}{2}dw = dx$

$\frac{2}{4}$

$\int \frac{2}{2x-1}\,dx + \int \frac{3}{2x-4}\,dx$

$= 2\left(\tfrac{1}{2}\right)\ln|2x-1| + 3\left(\tfrac{1}{2}\right)\ln|2x-4| + C$

$= \ln|2x-1| + \frac{3}{2}\ln|2x-4| + C$

Note: $= \ln|2x-1| + \frac{3}{2}\ln|2(x-2)| + C$

$= \ln|2x-1| + \frac{3}{2}\left[\ln 2 + \ln|x-2|\right] + C$

$= \ln|2x-1| + \frac{3}{2}\ln|x-2| + \underbrace{\frac{3}{2}\ln 2}_{\text{const}} + C$

$= \ln|2x-1| + \frac{3}{2}\ln|x-2| + C$

Both fine!

CASE 3: (CASE 1)(CASE 2)

eg
$$\frac{2x^3+5}{(x+1)^3(x+2)(x+3)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3} + \frac{D}{x+2} + \frac{E}{x+3}$$

repeated distinct

num deg < 5

eg
$$\frac{5x}{(x+2)^3(x-1)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$$

More cases: denom can't factor fully ex x^3+1
In book, <u>not</u> in syllabus

WHAT IF DEG OF NUM $\geq$ DEG OF DENOM

Method only works if deg num < deg denom

Use: Polynomial Long Division

If you have a method you like, use it!

i'll show one method (of many)

Analogy: $\frac{13}{3} = \frac{12+1}{3} = \frac{4\cdot\cancel{3}}{\cancel{3}} + \frac{1}{3} = 4 + \frac{1}{3}$

keep frac same

separate out lgst multiple of denom in num

cancel!

$$\frac{8x^2 + 22x + 23}{2x+3} = \frac{8x^2+12x+10x+23}{2x+3} = \frac{8x^2+12x}{2x+3} + \frac{10x+23}{2x+3}$$

22x

deg of num ≥ deg of denom

Find multiple of denom that will "take care of" highest term, $8x^2$

$4x(2x+3) = 8x^2+12x$

separated multiple of denom

fraction same

$$= \frac{4x\cancel{(2x+3)}}{\cancel{2x+3}} + \frac{10x+23}{2x+3}$$

$$= 4x + \frac{10x+23}{2x+3}$$

$$4x + \frac{10x+23}{2x+3} = 4x + \frac{10x+15+8}{2x+3} = 4x + \frac{10x+15}{2x+3} + \frac{8}{2x+3}$$

deg of num ≥ deg of denom

Get rid of 10x using multiple of 2x+3:

$5(2x+3) = 10x+15$

separate multiple of denom

No change in fraction

$$= 4x + \frac{5\cancel{(2x+3)}}{\cancel{2x+3}} + \frac{8}{2x+3}$$

$$= 4x + 5 + \frac{8}{2x+3}$$

deg of num < deg of denom

Now: $\displaystyle\int \frac{8x^2+22x+23}{2x+3}\,dx$

$$= \int 4x+5+\frac{8}{2x+3}\,dx = \boxed{2x^2+5x+4\ln|2x+3|+C}$$

If deg of num ≥ deg of denom

1. long division ← video on section's webpage week 7
2. method of partial fractions

Ch 7.7 : Numerical Integration

Approximate value of a definite integral in a way that we can bound our error.

Absolute error: $|\text{exact} - \text{approx}|$

Relative error: $\dfrac{|\text{exact} - \text{approx}|}{|\text{exact}|}$

CASE 1: Actual 500
Approx 495

Absolute error: 5
Relative error: $\frac{5}{500} = 0.01 = 1\%$

CASE 2: Actual: 5
Approx: 10

Absolute error: 5
Relative error: $\frac{5}{5} = 1 = 100\%$

3 Methods:

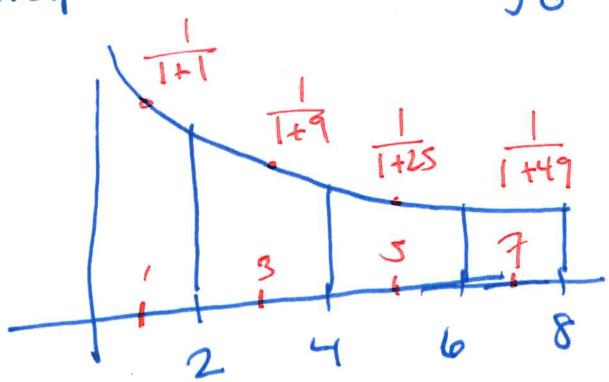
Midpoint Rule (Riemann Sum)

Trapezoid Rule (Sum of trapezoids, not rectangles)

Simpson's Rule (parabolas instead of lines)

Error bounds given p 565 in text – do not need to memorize, will be given

Midpt: approx $\int_0^8 \frac{1}{1+x^2}\,dx$, $n=4$ $\Delta x = \frac{8-0}{4} = 2$



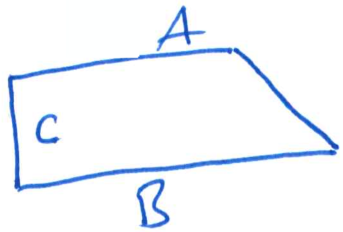
Approx: $(2)\left(\frac{1}{1+1}\right) + 2\left(\frac{1}{1+9}\right) + (2)\left(\frac{1}{1+25}\right) + 2\left(\frac{1}{1+49}\right)$

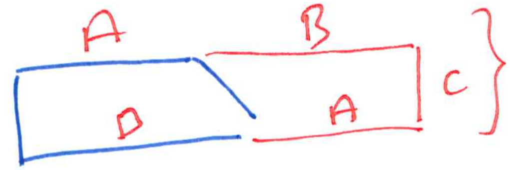
≈ 1.3

Notice: $\int_0^8 \frac{1}{1+x^2}\,dx = \arctan 8$

what is this #? 1? 0? 3/2?

Trapezoid:





$C(A+B)$

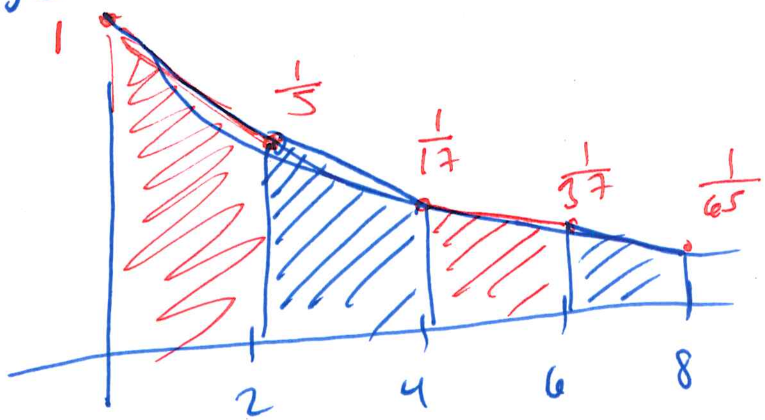
Area: $\frac{1}{2}C(A+B)$

$= \frac{1}{2}(\text{base})(\text{sum of heights})$

left

$$\int_0^8 \frac{1}{1+x^2}\,dx \approx \cancel{\tfrac{1}{2}}(\cancel{2})\left(1+\frac{1}{5}\right) + \cancel{\tfrac{1}{2}}(\cancel{2})\left(\frac{1}{5}+\frac{1}{17}\right) + \cancel{\tfrac{1}{2}}(\cancel{2})\left(\frac{1}{17}+\frac{1}{37}\right) + \cancel{\tfrac{1}{2}}(\cancel{2})\left(\frac{1}{37}+\frac{1}{65}\right)$$

right



$$\int_a^b f(x)\,dx \approx \frac{1}{2}\Delta x\, f(x_0) + \sum \Delta x\, f(x_i) + \frac{1}{2}\Delta x\, f(x_n)$$