Algebra Time

- Completing the square
- Polynomial long division + method of partial fractions

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ex $\int \frac{1}{\sqrt{1-x^2}}\,dx$

If you don't remember this is arcsine, use trig sub: $x=\sin\theta$: then

$\sqrt{1-x^2}=\sqrt{1-\sin^2\theta}=\sqrt{\cos^2\theta}$

$=\cos\theta$

ex $\int \frac{1}{\sqrt{1-(x+5)^2}}\,dx$ is the same as $\int \frac{1}{\sqrt{-x^2-10x-24}}\,dx$

Similarly: $(x+5)=\sin\theta$

In this form, choosing a substitution harder

ex $\int \left(\frac{3}{x}+\frac{4}{x^2}+\frac{1}{x-1}\right)dx$ is the same as $\int \frac{4x^2+x-4}{x^2(x-1)}\,dx$

$=3\ln|x|-\frac{4}{x}+\ln|x-1|+C$

In this form, difficult to evaluate

Completing the Square

$$\underbrace{ax^2+bx+c}_{\text{three terms}} = \underbrace{(ex+f)^2+d}_{\text{two terms}}$$

d ↑ const

$1-\sin^2\theta =$
$1+\tan^2\theta =$
$\sec^2\theta - 1 =$
(two terms)

Recall:

$$(x+a)^2 = \underbrace{x^2+2ax}\ +a^2$$

eg $\underbrace{x^2+6x}_{x^2+2ax+\cdots} + 1$ ← constant can fix later

$a=3$

$(x+3)^2 = \underbrace{x^2+6x}+9$

$$= \underbrace{x^2+6x+\overbrace{9}^{=0}-9}+1$$

$$= (x+3)^2 - 9 + 1$$

$$= \boxed{(x+3)^2-8}$$

eg $\displaystyle\int \frac{1}{\sqrt{x^2+6x+1}}\,dx = \int \frac{1}{\sqrt{(x+3)^2-8}}\,dx$

use a secant sub

$x+3 = \sqrt{8}\sec\theta$ (etc)

ID: $8\sec^2\theta - 8 = 8\tan^2\theta$

have $\boxed{(x+3)^2 - 8}$

So: $(x+3)^2 = 8\sec^2\theta$

$x+3 = \sqrt{8}\sec\theta$

eg $\quad \underbrace{x^2 - x} + 5 = x^2 - x + \frac{1}{4} - \frac{1}{4} + 5$

$\underbrace{x^2 + 2ax} + a^2$

$a = -1/2$

Square part: $(x - 1/2)^2 = x^2 + 2(x)(-1/2) + \frac{1}{4}$

$= (x^2 - x + \frac{1}{4})$

$= (x - 1/2)^2 - \frac{1}{4} + 5$

$= (x - 1/2)^2 + \frac{19}{4}$

Use: $\int \sqrt{x^2 - x + 5}\, dx = \int \sqrt{(x - 1/2)^2 + \frac{19}{4}}\, dx$

Use identify: $\frac{19}{4} + \frac{19}{4}\tan^2\theta = \frac{19}{4}\sec^2\theta$

Have : $\frac{19}{4} + (x - 1/2)^2$

sub : $\frac{19}{4}\tan^2\theta = (x - 1/2)^2$

$\frac{\sqrt{19}}{2}\tan\theta = x - 1/2$ substitution

Polynomial Long Division

Goal: polynomials $p(x)$, $q(x)$,

degree of $p \geq$ deg of q

$$\frac{p(x)}{q(x)} \dashrightarrow f(x) + \frac{g(x)}{q(x)}$$

$f(x)$, $g(x)$ poly

deg g < deg q

Compare: $\frac{13}{3} = \frac{12+1}{3} = \frac{12}{3} + \frac{1}{3} = \frac{4 \cdot 3}{3} + \frac{1}{3} = 4 + \frac{1}{3}$

top > bottom

$\frac{1}{3}$: top < bottom

Different methods of poly long division

Any is fine! If you already have a method - use that!

$$\frac{13}{3} = \frac{12+1}{3} = \frac{4\cdot 3}{3} + \frac{1}{3} = 4 + \frac{1}{3}$$

Separated out multiple of denomin (this will cancel later)

split numerator

cancel

$$\frac{8x^2+22x+23}{2x+3} = \frac{8x^2+12x\overbrace{-12x+22x}^{10x}+23}{2x+3}$$

Deg of num: 2
Deg of denom: 1

Denom: $\underbrace{2x}+3$
Highest power term in num: $\underline{8x^2}$

Note: $4x(2x+3) = 8x^2+12x$

$$= \frac{8x^2+12x}{2x+3} + \frac{10x+23}{2x+3} = \frac{4x\cancel{(2x+3)}}{\cancel{2x+3}} + \frac{10x+23}{2x+3} = 4x + \frac{10x+23}{2x+3}$$

Highest power in num: 10x
Denom: 2x+3

deg of num = deg of denom

Note: $5(2x+3) = 10x+15$

$$= 4x + \frac{10x+15-15+23}{2x+3} = 4x + \frac{10x+15}{2x+3} + \frac{8}{2x+3}$$

$$= 4x + \frac{5(\cancel{2x+3})}{\cancel{2x+3}} + \frac{8}{2x+3} = \boxed{4x + 5 + \frac{8}{2x+3}}$$

deg of num = 0
deg of denom = 1
(done)

Use: $\displaystyle\int \frac{8x^2+22x+23}{2x+3}\,dx$

$$= \int \left(4x + 5 + \frac{8}{2x+3}\right) dx = \boxed{2x^2 + 5x + 4\ln|2x+3| + C}$$

$u = 2x+3$
$du = 2\,dx$
$\frac{1}{2}du = dx$

$$\int \frac{8}{2x+3}\,dx = \int \frac{8}{u}\cdot\frac{1}{2}\,du$$

$$= \int 4\cdot\frac{1}{u}\,du = 4\ln|u| + C$$

$$= 4\ln|2x+3| + C$$