Ch 7.4 : Trigonometric Substitution

Warmup: evaluate $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,dx = \int_3^7 \frac{1}{\cancel{\sqrt{(x+1)^2}}}\,dx$ (*)

$$\int_3^7 \frac{1}{|x+1|}\,dx = \int_3^7 \frac{1}{x+1}\,dx$$

$$= \ln|x+1|\Big|_3^7 = \ln 8 - \ln 4 = \ln\frac{8}{4} = \boxed{\ln 2}$$

(*) $\sqrt{\text{quadratic}} = \cancel{\sqrt{(\text{something})^2}} = |\text{something}|$

cancel off root

ex $\int \frac{1}{\sqrt{x^2+1}}\,dx$

If I could write

$x^2+1 = [f(x)]^2$

then:

$\sqrt{x^2+1} = \cancel{\sqrt{\vphantom{x}}}\,[f(x)]^{\cancel{2}}$

$= |f(x)|$

To write x^2+1 as a perfect square, we use substitution & trig identities

$\tan^2\theta + 1 = \sec^2\theta$

Let $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

Then:

and $\sqrt{x^2+1} = \sqrt{\tan^2\theta+1}$

$= \cancel{\sqrt{\sec^2\theta}} = |\sec\theta|$

Fine print: to do this, we restrict domain of θ

then: $\sec\theta > 0$ so $|\sec\theta| = \sec\theta$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sec\theta} \cdot \underbrace{\sec^2\theta\, d\theta}_{dx} = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$\sqrt{x^2+1} +$ x

$$= \boxed{\ln\left|\sqrt{x^2+1} + x\right| + C}$$

θ is a new variable I made up: should give answer in terms of x

Sub: $x = \tan\theta$

simplification: $\sqrt{x^2+1} = \sec\theta$

Note: $x = \tan\theta$

$\Rightarrow \theta = \arctan x$

$\Rightarrow \sec\theta = \sec(\arctan x)$ ← bad form trig (arctrig)

$\sec\theta = \sqrt{x^2+1}$ ← way way easier to evaluate & understand

Method (most ~~general~~ standard case)

1) Given integrand $\sqrt{(\text{quadratic})}$
want to cancel root

2) Choose a trig substitution so that
quadratic $\longrightarrow$ perfect square
using:
$$\left.\begin{aligned} 1-\sin^2\theta &= \cos^2\theta \\ 1+\tan^2\theta &= \sec^2\theta \\ \sec^2\theta - 1 &= \tan^2\theta \end{aligned}\right\}$$

eg. $\sqrt{x^2-1}$
use $x = \sec\theta$
$\sqrt{x^2-1} = \sqrt{\sec^2\theta - 1}$
$= \sqrt{\tan^2\theta} = \tan\theta$

3) Trig integral — maybe simplify, then 7.3 techniques to antidiff.

4) Get answer in terms of x: sometimes work already done;
other method: use a triangle

Useful Facts:

$\sin(2x) = 2\sin x \cos x$

$\cos(2x) = \cos^2 x - \sin^2 x$

$\int \csc x \, dx = -\ln|\csc x + \cot x| + C$

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ex $\int \frac{dx}{(1+x^2)^{3/2}}$

Note: $(1+x^2)^{3/2} = \sqrt{1+x^2}^{\,3}$

Choose sub:

$x = \sin\theta$: $1+x^2 = 1+\sin^2\theta$ not a thing

$x = \tan\theta$: $1+x^2 = 1+\tan^2\theta = \sec^2\theta$

If we use $x = \tan\theta$:

$dx = \sec^2\theta \, d\theta$

$(1+x^2)^{3/2} = (1+\tan^2\theta)^{3/2}$

$= [(\sec\theta)^2]^{3/2} = (\sec\theta)^3$

$$\int \frac{1}{(1+x^2)^{3/2}}\,dx = \int \frac{1}{\sec^3\theta}\,\underbrace{\sec^2\theta\,d\theta}_{dx}$$

$$= \int \frac{1}{\sec\theta}\,d\theta = \int \cos\theta\,d\theta = \sin\theta + C$$

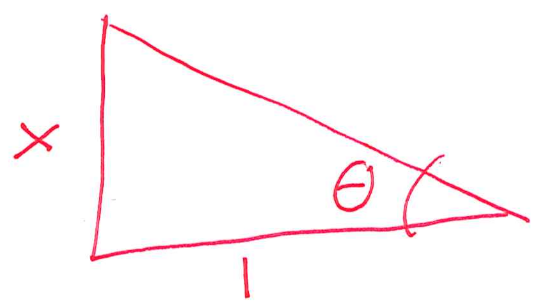
get in terms of x

Use a triangle to find $\sin\theta$ in terms of x.

Know: $x = \tan\theta$

$1+x^2 = \sec^2\theta$

$\frac{x}{1} = x = \tan\theta = \frac{\text{opp}}{\text{adj}}$



$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{1+x^2}}$

hyp: Pythagorean thm

$\text{hyp} = \sqrt{1+x^2}$

Final answer: $\sin\theta + C = \boxed{\frac{x}{\sqrt{1+x^2}} + C}$

ex

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$x = \sin\theta \qquad dx = \cos\theta\, d\theta$

then:

$\sqrt{1-x^2} = \sqrt{1-\sin^2\theta} = \cancel{\sqrt{\cos^{2}\theta}}$

$= \cos\theta$

$$= \int \frac{\sin^2\theta}{\cos\theta}\cdot \cos\theta\, d\theta$$

Identity: $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$$= \int \sin^2\theta\, d\theta \qquad (\text{Ch } 7.3)$$

$$= \int \tfrac{1}{2}(1-\cos 2\theta)\, d\theta = \tfrac{1}{2}\Big(\theta - \underbrace{\tfrac{1}{2}\sin(2\theta)}\Big) + C$$

$$= \tfrac{1}{2}\theta - \tfrac{1}{4}\big[\underbrace{2\sin\theta\cos\theta}\big] + C$$

$x = \sin\theta \longrightarrow \theta = \arcsin x$

Already found

$\cos\theta = \sqrt{1-x^2}$

$$= \boxed{\tfrac{1}{2}\arcsin x - \tfrac{1}{2}x\sqrt{1-x^2} + C}$$

ex To simplify $\sqrt{4x^2-1}$, which trig sub should I use?

Want to make $4x^2-1$ into $\sec^2\theta - 1$

$$4x^2-1 = \sec^2\theta - 1$$
$$4x^2 = \sec^2\theta$$
$$2x = \sec\theta$$
$$\boxed{x = \tfrac{1}{2}\sec\theta}$$

$$\sqrt{4x^2-1} = \sqrt{4\cdot\tfrac{1}{4}\sec^2\theta - 1} = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta}$$
$$= \tan\theta$$

ex To simplify $\sqrt{3+2x^2}$, what trig sub will work?

Identies: $3-3\sin^2\theta = 3\cos^2\theta$

$3+3\tan^2\theta = 3\sec^2\theta$ (*)

$3\sec^2\theta - 3 = 3\tan^2\theta$

Goal: $3+2x^2 \longrightarrow 3+3\tan^2\theta$

$2x^2 = 3\tan^2\theta$

$x^2 = \frac{3}{2}\tan^2\theta$

$$\boxed{x = \sqrt{\tfrac{3}{2}}\tan\theta}$$

$$\sqrt{3+2x^2} = \sqrt{3+2\cdot\tfrac{3}{2}\tan^2\theta} = \sqrt{3+3\tan^2\theta}$$

$$= \sqrt{3\sec^2\theta} = \sqrt{3}\cdot\sec\theta$$

ex $\int \frac{1}{\sqrt{3-x^2+2x}}\,dx$

$3-x^2$: 2 parts

$3-x^2+2x$: 3 parts

Identities: 2 parts

Complete the square!

$3-x^2+2x = -[\underbrace{x^2-2x}-3]$

$= -[\underbrace{x^2-2x+1}_{(x-1)^2}-4]$

$= -[(x-1)^2-4]$

$= 4-(x-1)^2$

2 parts!

$(a+b)^2 =$
$a^2+2ab+b^2$

$(x+b)^2 =$
$x^2+2bx+b^2$

match terms w/ x, fix constant

Identity: $1-\sin^2\theta = \cos^2\theta$

$4-4\sin^2\theta = 4\cos^2\theta$

Want: $4-(x-1)^2 = 4-4\sin^2\theta$

Want: $(x-1)^2 = 4\sin^2\theta$

$x-1 = 2\sin\theta$

$\boxed{x = 1+2\sin\theta}$

Then: $dx = 2\cos\theta\, d\theta$

$$\sqrt{3-x^2+2x} = \sqrt{4-(x-1)^2} = \sqrt{4-(\underbrace{2\sin\theta}_{x-1})^2}$$

completed □

$$= \sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta$$

$$\int \frac{1}{\sqrt{3-x^2+2x}}\, dx = \int \frac{1}{2\cos\theta} 2\cos\theta\, d\theta = \int 1\, d\theta$$

$$= \theta + C$$

$$= \boxed{\arcsin\left(\frac{x-1}{2}\right) + C}$$

$x-1 = 2\sin\theta$

$\frac{x-1}{2} = \sin\theta$

$\theta = \arcsin\left(\frac{x-1}{2}\right)$