Recall from Last Time

$$\underbrace{\int \sin^2 x \cos^3 x \, dx}_{\substack{\text{odd power of cosine:} \\ \text{use } u = \sin x \\ du = \cos x\, dx}} = \int \sin^2 x \underbrace{\cos^2 x}_{\substack{\downarrow \\ \text{change} \\ \text{to sines}}} \cdot \underbrace{\cos x \, dx}_{\substack{\text{save for} \\ du}} = \int \sin^2 x (1-\sin^2 x) \cos x \, dx$$

$$= \int u^2 (1-u^2)\, du = \int (u^2 - u^4)\, du$$

$$= \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \boxed{\frac{1}{3}\sin^3 x - \frac{1}{5}\sin^5 x + C}$$

$$\frac{d}{dx} \tan x = \sec^2 x \qquad \tan^2 x + 1 = \sec^2 x \qquad \int \tan x \, dx = \ln|\sec x| + C$$

$$\frac{d}{dx} \sec x = \sec x \tan x \qquad \int \sec x \, dx = \ln|\sec x + \tan x| + C$$

We can swap two secants for two tangents, and vice-versa, using: $\tan^2 x + 1 = \sec^2 x$

$\left.\begin{array}{l} u = \tan x \\ du = \sec^2 x\, dx \end{array}\right\}$ Save $\sec^2 x$ for du. Convert remaining secants $\longrightarrow$ tangents

Power of $\boxed{\text{secant}}$ should be $\boxed{\text{even}}$ (and at least 2)

$\left.\begin{array}{l} u = \sec x \\ du = \sec x \tan x\, dx \end{array}\right\}$ Save $\sec x \tan x$ for du. Convert remaining tangents $\rightarrow$ secants

Power of $\boxed{\text{tangent}}$ should be $\boxed{\text{odd}}$ (and there's at least one secant) ex $\int \sec^3 x \tan^3 x \, dx$

power of sec is odd

" " tan " odd

use $u = \sec x$

$du = \sec x \tan x \, dx$

$= \int \underbrace{\sec^2 x}_{u^2} \tan^2 x \underbrace{\sec x \tan x \, dx}_{du}$

secants

$= \int \sec^2 x (\sec^2 x - 1) \underbrace{\sec x \tan x \, dx}$

$= \int u^2 (u^2 - 1) \, du = \int (u^4 - u^2) \, du = \frac{1}{5} u^5 - \frac{1}{3} u^3 + C$

$= \boxed{\frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C}$

ex $\displaystyle\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$

$= \boxed{\tan x - x + C}$

Ch 7.4 : trig substitution

ex $\displaystyle\int_3^7 \frac{1}{\sqrt{x^2+2x+1}}\,dx = \int_3^7 \frac{1}{\cancel{\sqrt{(x+1)^2}}}\,dx = \int_3^7 \frac{1}{x+1}\,dx$

$= \ln|x+1| \Big|_3^7 = \ln 8 - \ln 4 = \ln(8/4) = \boxed{\ln 2}$

Compare: $\displaystyle\int \frac{1}{\sqrt{x^2+2x+1}}\,dx$ vs $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$

$\cancel{\sqrt{(\ \ )^2}}$

Would like:

$$x^2+1 = (\text{something})^2$$

Then: $\sqrt{x^2+1} = \cancel{\sqrt{(\text{something})^{\cancel{2}}}} = |\text{something}|$

($\sqrt{\ }$ gone)

Hiccup: $x^2+1 \neq (ax+b)^2$

Use substitution $x = \tan\theta$

Then: $\sqrt{x^2+1} = \sqrt{\tan^2\theta+1} = \cancel{\sqrt{\sec^{2}\theta}}$

$= |\sec\theta| = \sec\theta$ (2)

for 7.4, we'll only consider positive case

(1) $x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{1}{\sqrt{x^2+1}}\,dx = \int \frac{1}{\sec\theta}\sec^2\theta\, d\theta$$

$$= \int \sec\theta\, d\theta = \ln|\underset{(2)}{\sec\theta} + \underset{(1)}{\tan\theta}| + C = \boxed{\ln|\sqrt{x^2+1} + x| + C}$$

Method of Trig Substitution:

$\sqrt{(\text{quadratic})}$

(choose wisely)

1) Simplify a square root using trig substitution

2) evaluate resulting integral (methods from 7.3, maybe after simplifying)

3) get back original variable (often this can be done using previous work; otherwise can use a triangle)

Useful:

$1-\sin^2\theta = \cos^2\theta \longrightarrow \sqrt{1-\sin^2\theta} = \sqrt{\cos^2\theta} = \cos\theta$

$1+\tan^2\theta = \sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$\sin(2x) = 2\sin x\cos x$

$\cos(2x) = \cos^2 x - \sin^2 x$

$\int \csc x\, dx = -\ln|\csc x + \cot x| + C$

ex $\displaystyle\int \frac{1}{(1+x^2)^{3/2}}\,dx$

Recall:

$(1+x^2)^{3/2} = \left(\sqrt{1+x^2}\right)^3$

We want a trig sub: $\underbrace{1+x^2}_{\text{const + fcn}} \longrightarrow (\text{something})^2$

const − fcn [$1-\sin^2\theta$

const + fcn [$1+\tan^2\theta$

fcn − const [$\sec^2\theta - 1$

WANT: $1+x^2 \longrightarrow 1+\tan^2\theta$

$= \sec^2\theta$

use: $x = \tan\theta$

Then: $\sqrt{1+x^2} = \sqrt{1+\tan^2\theta} = \cancel{\sqrt{\sec^2\theta}}$

$= \sec\theta$

$dx = \sec^2\theta\, d\theta$

$$\int \frac{1}{(1+x^2)^{3/2}}\,dx = \int \frac{1}{\sqrt{1+x^2}^{\,3}}\,dx$$

$\sec\theta = \frac{1}{\cos\theta}$

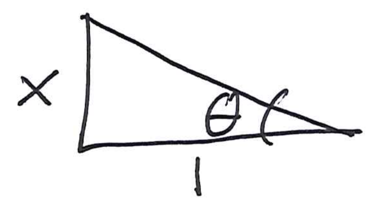
$$= \int \frac{1}{(\sec\theta)^3}\sec^2\theta\,d\theta = \int \frac{1}{\sec\theta}\,d\theta$$

$$= \int \cos\theta\,d\theta = \sin\theta + C$$

Change $\theta \to x$
(using the appropriate identity)

Use a triangle

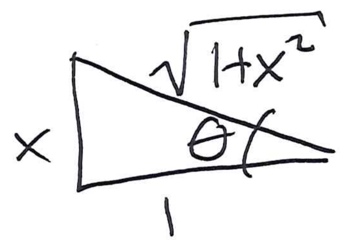
$\frac{x}{1} = x = \tan\theta = \frac{opp}{adj}$



$\tan\theta = x$

hyp (Pythagorean)

$\sqrt{1+x^2}$



$\sin\theta = \frac{opp}{hyp} = \frac{x}{\sqrt{1+x^2}}$

$$= \boxed{\frac{x}{\sqrt{1+x^2}} + C}$$

ex $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

① Choose a trig sub:

$x=\sin\theta \rightarrow 1-x^2=1-\sin^2\theta=\cos^2\theta$

or

$x=\sec\theta \rightarrow 1-x^2=1-\sec^2\theta$ not a thing

or

$x=\tan\theta \rightarrow 1-x^2=1-\tan^2\theta$ not a thing

(1) $x=\sin\theta \rightarrow \theta=\arcsin x$

$\left.\begin{aligned}\sqrt{1-x^2}&=\sqrt{1-\sin^2\theta}\\&=\cancel{\sqrt{\cos^{2}\theta}}=\cos\theta\end{aligned}\right\}$ (2)

$dx=\cos\theta\, d\theta$

$$\int\frac{x^2}{\sqrt{1-x^2}}\,dx=\int\frac{\sin^2\theta}{\cos\theta}\underbrace{\cos\theta\, d\theta}_{dx}=\int\sin^2\theta\, d\theta$$

$$=\int\frac{1-\cos(2\theta)}{2}\,d\theta=\frac{1}{2}\left[\theta-\frac{1}{2}\sin(2\theta)\right]+C$$

$$=\frac{1}{2}\left[\theta-\underbrace{\sin\theta}_{(1)}\underbrace{\cos\theta}_{(2)}\right]+C$$

$$=\boxed{\frac{1}{2}\left[\arcsin x-x\sqrt{1-x^2}\right]+C}$$

Could write:

$\cos\theta=\cos(\arcsin x)$

bad practice! almost impossible to evaluate like this ex $\int \frac{\sqrt{2x^2-3}}{x}\,dx$

$1-\sin^2\theta = \cos^2\theta$

$3-3\sin^2\theta = 3\cos^2\theta$

$1+\tan^2\theta = \sec^2\theta$

$3+3\tan^2\theta = 3\sec^2\theta$

$\sec^2\theta - 1 = \tan^2\theta$

$3\sec^2\theta - 3 = 3\tan^2\theta$

$$= \int \frac{\sqrt{3}\tan\theta}{\cancel{\sqrt{3}/\sqrt{2}\,\sec\theta}} \underbrace{\frac{\sqrt{3}}{\sqrt{2}}\cancel{\sec\theta}\tan\theta\,d\theta}_{dx}$$

$$= \sqrt{3}\int \tan^2\theta\,d\theta$$

$$= \sqrt{3}\int (\sec^2\theta - 1)\,d\theta$$

$$= \sqrt{3}\left[\tan\theta - \theta\right] + C$$

$$= \sqrt{2x^2-3} - \sqrt{3}\,\theta + C$$

$$= \boxed{\sqrt{2x^2-3} - \sqrt{3}\,\mathrm{arcsec}\left(\sqrt{\tfrac{2}{3}}\,x\right) + C}$$

HAVE: $2x^2-3$

WANT: $3\sec^2\theta - 3$

$$2x^2 - 3 = 3\sec^2\theta - 3$$

$$2x^2 = 3\sec^2\theta$$

$$\sqrt{2}\,x = \sqrt{3}\sec\theta$$

$$\boxed{x = \sqrt{3/2}\,\sec\theta}$$

$dx = \sqrt{3/2}\,\sec\theta\tan\theta\,d\theta$

$x\sqrt{2/3} = \sec\theta$

so $\theta = \mathrm{arcsec}\left(x\sqrt{2/3}\right)$

Check x is nice:

$$\sqrt{2x^2-3} = \sqrt{2\left(\tfrac{3}{2}\sec^2\theta\right) - 3} =$$

$$\sqrt{3\sec^2\theta - 3} = \sqrt{3\tan^2\theta} = \sqrt{3}\tan\theta$$

$\sqrt{3}\tan\theta = \sqrt{2x^2-3}$