Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\sin^2 x = \frac{1-\cos(2x)}{2} \qquad \cos^2 x = \frac{1+\cos(2x)}{2}$$

ex

$$\int \sin^2 x \cos^2 x \, dx = \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2} \, dx$$

$$= \int \frac{1-\cos^2(2x)}{4} \, dx = \int \left[\frac{1}{4} - \frac{1}{4}\cos^2(2x)\right] dx$$

$$= \frac{1}{4}x - \frac{1}{4}\int \cos^2(2x) \, dx = \frac{1}{4}x - \frac{1}{4}\int \frac{1+\cos(4x)}{2} \, dx$$

$$= \frac{1}{4}x - \int \left[\frac{1}{8} + \frac{1}{8}\cos(4x)\right] dx = \underbrace{\frac{1}{4}x - \frac{1}{8}x} - \int \frac{1}{8}\cos(4x) \, dx$$

$$u = 4x$$

$$du = 4\,dx$$

$$\frac{1}{4}du = dx$$

$$= \frac{1}{8}x - \int \frac{1}{8} \cdot \cos(u) \cdot \frac{1}{4} \, du$$

$$= \boxed{\frac{1}{8}x + \frac{-1}{32}\sin(4x) + C}$$

Method:

$\left.\begin{array}{l} \sin^2 \to \cos \\ \cos^2 \to \cos \end{array}\right\}$ $\cos^2 \to \cos$

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ex $\int \sin x \underbrace{\cos x\, dx}_{du}$

$u = \sin x$
$du = \cos x\, dx$

$$\int u\, du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\sin^2 x + C}$$

ex $\int \sin^{10} x \underbrace{\cos x\, dx}$

$u = \sin x$
$du = \cos x\, dx$

$$= \int u^{10}\, du = \frac{1}{11}u^{11} + C = \boxed{\frac{1}{11}\sin^{11} x + C}$$

(ex) $\int \sin^4 x \cdot \cos^3 x \, dx$

$u = \sin x$
$du = \cos x \, dx$

$= \int \underbrace{\sin^4 x}_{u^4} \cdot \underbrace{\cos^2 x}_{?} \cdot \underbrace{\cos x \, dx}_{du}$

$\sin^2 x + \cos^2 x = 1$
So $\cos^2 x = 1 - \sin^2 x$

$= \int \underbrace{\sin^4 x}_{u^4} \underbrace{(1 - \sin^2 x)}_{(1 - u^2)} \underbrace{\cos x \, dx}_{du}$

$= \int u^4 (1 - u^2) \, du = \int (u^4 - u^6) \, du = \frac{1}{5} u^5 - \frac{1}{7} u^7 + C$

$= \boxed{\frac{1}{5} \sin^5 x - \frac{1}{7} \sin^7 x + C}$

ex $\int \sin^2 x \cos^5 x \, dx$

$u = \sin x$
$du = \cos x \, dx$

$$= \int \underbrace{\sin^2 x}_{u^2} \cdot \underbrace{\cos^4 x}_{?} \cdot \underbrace{\cos x \, dx}_{du}$$

$\cos^4 x = [\cos^2 x]^2$
$= [1 - \sin^2 x]^2$

$$= \int \sin^2 x \, [1 - \sin^2 x]^2 \underbrace{\cos x \, dx}$$

$$= \int u^2 (1-u^2)^2 \, du = \int u^2 (1 - 2u^2 + u^4) \, du$$

$$= \int (u^2 - 2u^4 + u^6) \, du = \underline{\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C}$$

$$= \boxed{\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C}$$

We used $u = \sin x$
$du = \cos x \, dx$

- ① Saved <u>one</u> cosine for du
- ② Changed remaining cosines → sines using $\cos^2 x = 1 - \sin^2 x$

To avoid $\sqrt{\ }$: change cosines in <u>pairs</u>

So it was important we had an even # of cosines remaining

All together, started with [even # + 1] so: <u>odd</u> # of cosines.

$$\int \sin^3 x \cos^2 x \, dx$$

Power of cosine <u>even</u>

Power of sine <u>odd</u>

$$= \int \underbrace{\sin^2 x}_{?} \, \underbrace{\cos^2 x}_{u^2} \, \underbrace{\sin x \, dx}_{-du}$$

$$\boxed{\begin{aligned} u &= \cos x \\ du &= -\sin x \, dx \end{aligned}}$$

① Save one sine for du

② Change remaining sines into cosines

$\sin^2 x = 1 - \cos^2 x$

odd power of sine

$$= \int \underbrace{(1-\cos^2 x)}_{(1-u^2)} \, \underbrace{\cos^2 x}_{u^2} \, \underbrace{\sin x \, dx}_{-du}$$

$$= \int (1-u^2) u^2 (-1) \, du$$

$$= \int u^2 (u^2 - 1) \, du = \int (u^4 - u^2) \, du = \frac{1}{5} u^5 - \frac{1}{3} u^3 + C$$

$$= \boxed{\frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C}$$

Which Substitution?

$u=\sin$ / $u=\cos$ / half-angle

$\int \sin^2 x \cos^5 x \, dx$ $\quad u=\sin x$

$\int \sin^4 x \cos^2 x \, dx$ $\quad$ half-angle

$\int \sin^3 x \cos x \, dx$ $\quad u=\sin x$ will work (mildly easier)

$u=\cos x$ " "

ex $\int \sin^{0.75} x \cos^3 x \, dx$

Fact: 0.75 is neither odd nor even

3 is odd

$= \int \sin^{0.75} x \cos^2 x \underbrace{\cos x \, dx}$

$u = \sin x$

$du = \cos x \, dx$

① save one cosine for du

② remaining cos $\rightarrow$ sin

$= \int \sin^{0.75} x (1 - \sin^2 x) \underbrace{\cos x \, dx}$

$= \int u^{0.75} (1 - u^2) \, du$

$= \int (u^{0.75} - u^{2.75}) \, du = \frac{u^{1.75}}{1.75} - \frac{u^{3.75}}{3.75} + C$

$= \boxed{\frac{1}{1.75} \sin^{1.75} x - \frac{1}{3.75} \sin^{3.75} x + C}$

Recall: $\tan^2 x + 1 = \sec^2 x$

$\frac{d}{dx}(\tan x) = \sec^2 x$ $\quad$ $\frac{d}{dx}(\sec x) = \sec x \tan x$

$\int \tan x \, dx = \ln|\sec x| + C$ $\quad$ $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

$$\int \tan^3 x \sec^3 x \, dx$$

Option 1: $u = \tan x$
$du = \sec^2 x \, dx$

① reserve $\sec^2 x$ for du

② remaining sec → tan

Option 2: $u = \sec x$
$du = \sec x \tan x \, dx$

① reserve $\sec x \tan x$ for du

② convert remaining tan → sec

Option 1

$u = \tan x$
$du = \sec^2 x\,dx$

$$\int \underbrace{\tan^3 x}_{u^3} \; \sec x \; \underbrace{\sec^2 x\,dx}_{du}$$

$\sec x \downarrow$?

$\pm\sqrt{\cdots}$ ✗

Option 2

$u = \sec x$
$du = \sec x \text{ tangent } x\,dx$ } odd # tangents

even # tangents   one tan here

$$\int \tan^2 x \; \underbrace{\sec^2 x}_{u^2} \; \underbrace{\sec x \tan x\,dx}_{du}$$

$\tan^2 x = \sec^2 x - 1$

$$\int (\sec^2 x - 1)\sec^2 x \; \underbrace{\sec x \tan x\,dx}$$

$$= \int (u^2 - 1)u^2\,du = \int (u^4 - u^2)\,du = \boxed{\frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C}$$