Recall: Substitution Rule
(inverse of the chain rule)

ex $\displaystyle\int \sqrt{x^3 - \ln x}\,\left(3x^2 - \frac{1}{x}\right) dx$

The derivative of $(x^3 - \ln x)$ is multiplied to the rest of the integrand. We see this behaviour when we use the chain rule to differentiate.

So, let $u = x^3 - \ln x$. Then $du = \left(3x^2 - \frac{1}{x}\right) dx$

With these conventions:

$$\int \sqrt{x^3 - \ln x}\,\left(3x^2 - \frac{1}{x}\right) dx = \int \sqrt{u}\, du$$

Now, we can evaluate more easily.

$$= \int u^{1/2}\, du = \frac{2}{3} u^{3/2} + C = \boxed{\frac{2}{3}(x^3 - \ln x)^{3/2} + C}$$

Checking our result: $\displaystyle\frac{d}{dx}\left(\frac{2}{3}(x^3 - \ln x)^{3/2} + C\right) = \frac{2}{3}\cdot\frac{3}{2}(x^3 - \ln x)^{1/2} \cdot \left(3x^2 - \frac{1}{x}\right)$

$$= \sqrt{x^3 - \ln x}\,\left(3x^2 - \frac{1}{x}\right)$$

Ch 7.3: trig integrals

No new concepts:
applying substitution rule
to very specific functions

These functions show up in Ch 7.4

Remember:

$$\boxed{\begin{aligned} \sin^2 x + \cos^2 x &= 1 \\ \tan^2 x + 1 &= \sec^2 x \\ \sin^2 x &= \frac{1-\cos 2x}{2} \\ \cos^2 x &= \frac{1+\cos 2x}{2} \end{aligned}}$$

$$\frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\left(\frac{\sin x}{\cos x}\right)^2 + \frac{\cos^2 x}{\cos^2 x} = \sec^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

(ex) $\int \underset{u}{\sin x}\, \underbrace{\cos x\, dx}_{du}$ $u = \sin x$
$du = \cos x\, dx$

$\int u\, du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}\sin^2 x + C}$

(ex) $\int \underbrace{\sin^{10} x}_{u^{10}}\, \underbrace{\cos x\, dx}_{du}$ $u = \sin x$
$du = \cos x\, du$

$= \int u^{10}\, du = \frac{1}{11}u^{11} + C = \boxed{\frac{1}{11}\sin^{11} x + C}$

(ex) $\int \sin^4 x \cos^5 x\, dx$ $u = \sin x$
$du = \cos x\, dx$

$\sin^2 x + \cos^2 x = 1$
so: $\cos^2 x = 1 - \sin^2 x$

$= \int \underbrace{\sin^4 x}_{u^4} \cdot \underbrace{\cos^4 x}_{?} \cdot \underbrace{\cos x\, dx}_{du}$

$= \int \sin^4 x \cdot (\cos^2 x)^2 \cos x\, dx$

$= \int \underbrace{\sin^4 x\, (1 - \sin^2 x)^2}_{u^4(1-u^2)^2}\, \underbrace{\cos x\, dx}_{du}$

$$= \int u^4(1-u^2)^2\,du = \int u^4(1-2u^2+u^4)\,du$$

$$= \int (u^4-2u^6+u^8)\,du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$$

$$= \boxed{\frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C}$$

Generalize:

$$\int \sin^a x \cdot \cos^b x \, dx$$

To use
$u = \sin x$
$du = \cos x \, dx$:

I NEED:

- ONE cos for $du = \cos x \, dx$
- All other cosines $\longrightarrow$ sines
  so I can write in terms of u

  To do that, we use $\cos^2 x = 1 - \sin^2 x$
  To avoid $\sqrt{\ }$: we change
  <u>two</u> cosines at a time

- All together, power of cosine (s):
  $1 + (\text{even})$: <u>must be odd</u>

ex $\int \sin^{2.71} x \cos^5 x \, dx$

2.71 neither even nor odd
5 odd

$u = \sin x$
$du = \cos x \, dx$

Odd power of cosine:
save one for du
left w/ even power
change $\cos^2 x \to 1 - \sin^2 x$.

$$= \int \sin^{2.71} x \cdot (\cos^2 x)^2 \cos x \, dx$$

$$= \int \sin^{2.71} x \cdot (1 - \sin^2 x)^2 \underbrace{\cos x \, dx}$$

$$= \int u^{2.71} (1 - u^2)^2 \, du$$

$$= \int u^{2.71} (1 - 2u^2 + u^4) \, du = \int (u^{2.71} - 2u^{4.71} + u^{6.71}) \, du$$

$$= \frac{u^{3.71}}{3.71} - \frac{2u^{5.71}}{5.71} + \frac{u^{7.71}}{7.71} + C$$

$$= \boxed{\frac{1}{3.71} \sin^{3.71} x - \frac{2}{5.71} \sin^{5.71} x + \frac{1}{7.71} \sin^{7.71} x + C}$$

ex $\int \sin^3 x \cos^2 x \, dx$

$u = \cos x \, dx$

$du = -\sin x \, dx$

$= \int \cos^2 x \cdot \sin^2 x \cdot \sin x \, dx$

$= \int \cos^2 x (1-\cos^2 x) \cdot \sin x \, dx$

(no $\sqrt{\ }$)

$= \int u^2 (1-u^2)(-1) \, du$

$= \int u^2 (u^2 - 1) \, du$

$= \int (u^4 - u^2) \, du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C$

$= \boxed{\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C}$

Caution:

$u = \sin x$

$du = \cos x \, dx$

$\int \underbrace{\sin^3 x}_{u^3} \cdot \underbrace{\cos x}_{?} \cdot \underbrace{\cos x \, dx}_{du}$

$\cos^2 x = 1 - \sin^2 x$

$\cos x = \pm\sqrt{1-\sin^2 x}$

$\pm \int u^3 \sqrt{1-u^2} \, du$

$\sqrt{\ }$ are gross

Nicer: $u = \cos x$

$-du = \sin x \, dx$

$\sin^3 x = (\sin^2 x)\sin x$

$= \underbrace{(1-\cos^2 x)}_{1-u^2} \underbrace{\sin x}_{du}$

ex $\int \sin^3 x \cos^5 x \, dx$

$u = \sin x$

$du = \cos x \, dx$

$= \int \sin^3 x \cdot (\cos^2 x)^2 \cdot \cos x \, dx$

- Save one cosine (left w/ even #)
- Convert $\cos^2 x \to \sin^2 x$

$= \int \underbrace{\sin^3 x \, (1-\sin^2 x)^2}_{\text{in terms of } \sin x = u} \underbrace{\cos x \, dx}_{du}$

$= \int u^3 (1-u^2)^2 \, du$ (etc)

ex

$$\int \sin^2 x\,dx = \int \frac{1-\cos(2x)}{2}\,dx$$

(identity)

$$= \frac{1}{2}\int [1-\cos(2x)]\,dx$$

$$= \frac{1}{2}\left[x - \int \cos(2x)\,dx\right]$$

$$= \frac{1}{2}\left[x - \frac{1}{2}\sin(2x)\right] + C$$

$$= \boxed{\frac{1}{2}x - \frac{1}{4}\sin(2x) + C}$$

$$= \frac{1}{2}x - \frac{1}{2}\sin x\cos x + C$$

Cautionary Tale

$$\int (\sin x)^2\,dx = \frac{(\sin x)^3}{3\cos x}$$

Check: $$\frac{d}{dx}\left(\frac{(\sin x)^3}{3\cos x}\right) = \frac{3\cos x \cdot 3\sin^2 x \cos x + 3\sin^3 x \sin x}{9\cos^2 x}$$

quotient

(ex) $\int \sin^2 x \cos^2 x \, dx$

Recall:

$$\sin^2 x = \frac{1-\cos(2x)}{2}$$

$$\cos^2 x = \frac{1+\cos(2x)}{2}$$

$$= \int \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right) dx$$

Let $\theta = 2x$

$$= \frac{1}{4}\int \left[1-\cos^2(2x)\right] dx$$

$$\cos^2(2x) = \frac{1+\cos(4x)}{2}$$

$$= \frac{1}{4}\left[x - \int \cos^2(2x)\,dx\right]$$

$$= \frac{1}{4}\left[x - \int \frac{1+\cos(4x)}{2}\,dx\right]$$

Notice: no u-substitution

even powers of both $\sin x$ + $\cos x$

$$= \frac{1}{4}x - \frac{1}{8}\int (1+\cos(4x))\,dx$$

could have used $u=4x$

$$= \frac{1}{4}x - \frac{1}{8}\left[x + \frac{1}{4}\sin(4x)\right] + C$$

$$= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{32}\sin(4x) + C = \boxed{\frac{1}{8}x - \frac{1}{32}\sin(4x) + C}$$

Which Sub will Give Us Polynomial?

$$\int \sin^2 x \cos^3 x \, dx$$

$u = \sin x$
$du = \cos x \, dx$

$$\int \sin^5 x \cos^4 x \, dx$$

$u = \cos x$

$$\int \sin^6 x \cos^9 x \, dx$$

$u = \sin x$

$$\int \sin^3 x \cos^5 x \, dx$$

$u = \sin x$ & $u = \cos x$
<u>both work</u>

Table 7.2, p526 book:

$\int \sin^a x \cos^b x \, dx$

a odd ...

b odd ...

a, b both even ...

Table 7.3 p529 book

$\int \tan^a x \sec^b x \, dx$

Reduction formulas <u>not</u> in syllabus

<u>Recall</u>

$\frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\sec x) = \sec x \cdot \tan x$

$\int \tan x \, dx = \ln|\sec x| + C$

(new) $\int \sec x = \ln|\sec x + \tan x| + C$

To use $u = \tan x$
$du = \sec^2 x \, dx$

$$\int \tan^a x \sec^b x \, dx$$

- save 2 secants for du
- convert rest of secants → tangents (u) using $1 + \tan^2 x = \sec^2 x$

To avoid square roots, convert secants in pairs ($\sec^2 x$)

So: need even # left over

So: power of sec x should be even

ex
$$\int \tan^4 x \sec^4 x \, dx = \int \tan^4 x \sec^2 x \sec^2 x \, dx$$

$u = \tan x$
$du = \sec^2 x \, dx$

$$= \int \underset{u^4}{\tan^4 x} \underset{(1+u^2)}{(1 + \tan^2 x)} \underbrace{\sec^2 x \, dx}_{du}$$

$$= \int u^4 (1+u^2) \, du = \int (u^4 + u^6) \, du = \frac{1}{5} u^5 + \frac{1}{7} u^7 + C$$

$$= \boxed{\frac{1}{5} \tan^5 x + \frac{1}{7} \tan^7 x + C}$$

To use $u = \sec x$, $du = \sec x \tan x$ $\qquad \int \tan^a x \sec^b x \, dx$

- save $\boxed{\sec x \tan x}$ for du

(one tan)

- convert remaining $\boxed{\text{tangents}}$ to $\boxed{\text{secants}}$ using identity $\boxed{\tan^2 x = \sec^2 x - 1}$

To avoid square roots need $\boxed{\text{even}}$ number of $\boxed{\text{tangents}}$ remaining

(even left)

So: power of $\boxed{\tan}$ should be $\boxed{\text{odd}}$

ex $\int \underbrace{\sec^2 x}_{du} \tan^2 x \, dx$

$u = \tan x$
$du = \sec^2 x \, dx$

$= \int u^2 du = \frac{1}{3}u^3 + C = \boxed{\frac{1}{3}\tan^3 x + C}$

$u = \sec x$
$du = \sec x \tan x \, dx$

ex $\int \tan^3 x \sec x \, dx$

$= \int \tan^2 x \cdot \underbrace{\sec x \tan x \, dx}$

$= \int \underbrace{(\sec^2 x - 1)}_{u^2-1} \underbrace{\sec x \tan x \, dx}_{du} = \int (u^2 - 1) \, du$

$= \frac{1}{3}u^3 - u + C = \boxed{\frac{1}{3}\sec^3 x - \sec x + C}$

Choose your Substitution

$$\int \sec^4 x \tan^4 x\,dx$$

$u = \tan x$
(even power of sec)

$$\int \sec^4 x \tan^3 x\,dx$$

$u = \tan x$ ok
$u = \sec x$ ok

$$\int \sec x \tan^3 x\,dx$$

$u = \sec x$

$\int \sec x \tan^2 x\,dx$: ignoring these
(even tan, odd sec)

$$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx$$
$$= \boxed{\tan x - x + C}$$