Remember from Last Time:

Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

Shorthand for:

$$\int u(x) \cdot v'(x)\,dx = u(x)\,v(x) - \int v(x)\,u'(x)\,dx$$

Integrating the product of two functions:

- choose one as u
- choose the rest as dv
- u gets differentiated
- dv gets integrated

ex $\int x^3 \underbrace{\ln x}\, dx$

No obvious substitution

$u = \ln x \xrightarrow{\text{diff}} du = \frac{1}{x}\,dx$

$dv = x^3\,dx \xrightarrow[\text{antidiff}]{} v = \frac{1}{4}x^4$

Don't know $\int \ln x\,dx$
so $dv \neq \ln x\,dx$

$= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\,dx$

$= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3\,dx$

$= \boxed{\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C}$

ex $\int x^3(\ln x)^2\,dx$

$$u=(\ln x)^2 \xrightarrow{\text{diff}} 2\ln x\cdot\frac{1}{x}\,dx$$

$$dv = x^3\,dx \xrightarrow[\text{antidiff}]{} \frac14 x^4 = v$$

$$= \frac14 x^4\ln^2 x - \int \frac14 x^4\cdot 2\ln x\cdot\frac1x\,dx$$

$$= \frac14 x^4\ln^2 x - \frac12\int x^3\ln x\,dx$$

$$= \frac14 x^4\ln^2 x - \frac12\left[\frac14 x^4\ln x - \int \frac14 x^4\cdot\frac1x\,dx\right]$$

$$= \frac14 x^4\ln^2 x - \frac18 x^4\ln x + \frac18\int x^3\,dx$$

$$= \boxed{\frac14 x^4\ln^2 x - \frac18 x^4\ln x + \frac18\cdot\frac14 x^4 + C}$$

$$\int u\,dv = uv - \int v\,du$$

$$u \xrightarrow{\text{diff}} du$$

$$dv \xrightarrow[\text{antidiff}]{} v$$

$$\frac{x^4}{\cancel{x}} = x^3$$

This is progress: $\ln^2 x \to \ln x$

Sometimes you need to repeat

$u = \ln x \qquad du = \frac1x\,dx$

$dv = x^3\,dx \qquad v = \frac14 x^4$

ex $\int \ln x \, dx = \int \ln x \cdot 1 \, dx$

$u = \ln x \xrightarrow{\text{diff}} du = \frac{1}{x} dx$

$dv = 1 \, dx \xrightarrow[\text{antidiff}]{} v = x$

$= x\ln x - \int x \cdot \frac{1}{x} dx$

$= x\ln x - \int 1 \, dx$

$= \boxed{x\ln x - x + C}$

Check:

$\frac{d}{dx}(x\ln x - x + C) =$

$x \cdot \frac{1}{x} + \ln x \cdot 1 - 1 + 0 =$

$1 + \ln x - 1 = \boxed{\ln x}$ ✓

Inverse trig facts:

$$\left.\begin{aligned} \frac{d}{dx}(\arctan x) &= \frac{1}{1+x^2} \\ \frac{d}{dx}(\arcsin x) &= \frac{1}{\sqrt{1-x^2}} \\ \frac{d}{dx}(\arccos x) &= \frac{-1}{\sqrt{1-x^2}} \end{aligned}\right\}$$

"nice" in same way as $\frac{d}{dx}(\ln x) = \frac{1}{x}$

$$\int \arctan x \, dx = \int \arctan x \cdot 1 \, dx$$

$$= x \arctan x - \int \frac{x}{1+x^2} dx$$

$$= x \arctan x - \int \frac{1}{\omega} \cdot \frac{1}{2} d\omega$$

$$= x \arctan x - \frac{1}{2} \ln|\omega| + C$$

$$= \boxed{x \arctan - \frac{1}{2} \ln(1+x^2) + C}$$

IBP

$u = \arctan x$ $\quad du = \frac{1}{1+x^2} dx$

$dv = 1\, dx$ $\quad v = x$

Substitution

$\omega = 1+x^2$

$\frac{d\omega}{dx} = 2x$

$\frac{1}{2} d\omega = x\, dx$

$$\frac{x}{1+x^2} \neq \frac{x}{1} + \frac{x}{x^2} \quad : \quad \frac{1}{2} = \frac{1}{1+1} \neq \frac{1}{1} + \frac{1}{1} = 2$$

One more trick: no name

"integrating around in a circle"

only works when IBP changes integrand very little

ex $\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

$u = e^x \longrightarrow du = e^x dx$

$dv = \cos x \, dx \longrightarrow v = \sin x$

||

$u = e^x \longrightarrow du = e^x dx$

$dv = \sin x dx \longrightarrow -\cos x = v$

$e^x \sin x - \left[-e^x \cos x - \int e^x (-\cos x) dx \right]$

$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$

So: $\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx + C$

$+\int e^x \cos x \, dx \qquad\qquad +\int e^x \cos x \, dx$

$$2\int e^x \cos x \, dx = e^x \sin x + e^x \cos x + C$$

$$\boxed{\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C}$$

ex $\int e^{-x}\sin x\,dx =$

$u = e^{-x} \longrightarrow du = -e^{-x}\,dx$

$dv = \sin x\,dx \longrightarrow v = -\cos x$

$$-e^{-x}\cos x - \int e^{-x}\cos x\,dx$$

$u = e^{-x} \longrightarrow du = -e^{-x}\,dx$

$dv = \cos x\,dx \longrightarrow v = \sin x$

$$= -e^{-x}\cos x - \left[e^{-x}\sin x - \int -e^{-x}\sin x\,dx\right]$$

$$= -e^{-x}\cos x - e^{-x}\sin x - \cancel{\int e^{-x}\sin x\,dx} \qquad = \int e^{-x}\sin x\,dx$$

$$+\cancel{\int e^{-x}\sin x\,dx} \qquad +\int e^{-x}\sin x\,dx$$

$$-e^{-x}(\cos x + \sin x) = 2\int e^{-x}\sin x\,dx$$

So

$$\boxed{\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\cos x + \sin x) + C}$$

Definite integral
has bounds
area under curve
(#)

$\int_a^b f(x)\,dx$

Indefinite integral
no bounds

$\int f(x)\,dx$

family of functions
whose deriv is $f(x)$
$(F(x) + C)$

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ex $\int_0^{10} xe^x\,dx = xe^x \Big|_0^{10} - \int_0^{10} e^x\,dx$

$u = x \longrightarrow du = dx$

$dv = e^x dx \longrightarrow v = e^x$

$= (10e^{10} - 0e^0) - \int_0^{10} e^x\,dx$

$= 10e^{10} - [e^{10} - e^0] = \boxed{9e^{10} + 1}$

$e^2 = \frac{e^3}{e}$

$\boxed{e^0 = 1}$ $e = \frac{e^2}{e}$

$1 = \frac{e}{e} = e^0$

(ex) $\int_0^{2\pi} \underbrace{x^2}\underbrace{\sin x\,dx}$

$u = x^2 \longrightarrow du = 2x\,dx$

$dv = \sin x\,dx \longrightarrow v = -\cos x$

$= -x^2\cos x\Big|_0^{2\pi} - \int_0^{2\pi} -2x\cos x\,dx$

$= -(2\pi)^2\underbrace{\cos(2\pi)}_{1} + 0 + \int_0^{2\pi} 2x\cos x\,dx$

$x^2 \to x$: progress
do it again

$= -4\pi^2 + \int_0^{2\pi} 2x\cos x\,dx$

$u = 2x \longrightarrow du = 2\,dx$

$dv = \cos x\,dx \longrightarrow v = \sin x$

$= -4\pi^2 + \left[2x\sin x\Big|_0^{2\pi} - \int_0^{2\pi} 2\sin x\,dx \right]$

$= -4\pi^2 + 4\pi\underbrace{\cancel{\sin(2\pi)}}_{=0} - \cancel{2\cdot 0\sin 0} - \int_0^{2\pi} 2\sin x$

$= -4\pi^2 + 2\cos x\Big|_0^{2\pi}$

$= -4\pi^2 + \cancel{2\cos(2\pi)} - \cancel{2\cos(0)} = \boxed{-4\pi^2}$