More tools for antidifferentiating.

Ch 7.2 Integration by Parts

(not on next week's quiz)

Product rule, backwards

$$\frac{d}{dx}\Big(u(x)\,v(x)\Big) = u'(x)v(x) + u(x)v'(x) \quad \text{(Product Rule)}$$

$$\int \left[u'(x)v(x) + u(x)v'(x)\right] dx = u(x)v(x) + C$$

$$\int v(x)u'(x)\,dx + \int u(x)v'(x)\,dx = u(x)v(x) + C$$

$$\rightarrow \quad \int \underbrace{u(x)}\,\underbrace{v'(x)}\,dx = u(x)v(x) - \int \underbrace{v(x)}\,\underbrace{u'(x)}\,dx + C$$

Product of 2 functions

$u(x)$ gets diff'ed

$v'(x)$ gets antidiff'ed

$u(x)$ call u $v'(x)\,dx$ call dv

$u'(x)dx$ call du $v(x)$ call v (like in sub rule)

$$\boxed{\int u \cdot dv = uv - \int v\,du + C}$$ Integration by Parts

Goal: $u \xrightarrow{\text{diff}} du$ $\int v\,du$ easier

$dv \xrightarrow[\text{antidiff}]{} v$ (we hope)

No obvious substitution

ex $\int \underbrace{\ln x}_{u} \cdot \underbrace{x\,dx}_{dv}$

$u = \ln x \xrightarrow{\text{diff}} \frac{1}{x}dx = du$

$dv = x\,dx \xrightarrow[\text{antidiff}]{} \frac{1}{2}x^2 = v$

$$\int u\,dv = uv - \int v\,du$$

$$= \ln x \cdot \frac{1}{2}x^2 - \int \frac{1}{2}x^2 \cdot \frac{1}{x}\,dx$$

$$= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x\,dx$$

$$= \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$$

ex $\int x \sin x \, dx$

No obvious sub

Functions: x, $\sin x$

$$\int u\,dv = uv - \underbrace{\int v\,du}$$

Choices: (*) $u = x \xrightarrow{\text{diff}} du = dx$

$dv = \sin x \, dx \xrightarrow[\text{antidiff}]{} v = -\cos x$

$\int -\cos x \, dx$

$= x(-\cos x) - \int -\cos x \, dx$

$= \boxed{-x\cos x + \sin x + C}$

$u = \sin x \xrightarrow{\text{diff}} \cos x \, dx = du$

$dv = x\,dx \xrightarrow[\text{antidiff}]{} v = \frac{1}{2}x^2$

$\int \frac{1}{2}x^2 \cos x \, dx$

harder!

Check:

$$\frac{d}{dx}\left[\underbrace{-x\cos x} + \sin x + C\right]$$

$$= \underbrace{(-x)(-\sin x) + \cancel{\cos x}(-1)} + \cancel{\cos x}$$

$= x\sin x$ ✓

ex $\displaystyle\int \underbrace{(x+1)}_{u}\,\underbrace{e^x}_{dv}\,dx$

$= (x+1)e^x - \int e^x\,dx$

$= (x+1)e^x - e^x + C$

$= xe^x + e^x - e^x + C$

$= \boxed{xe^x + C}$

Another option:

$\int (x+1)e^x\,dx = \int (xe^x + e^x)\,dx$

$= \int xe^x\,dx + \int e^x\,dx$

$= \underbrace{\int xe^x\,dx}_{\text{Parts}} + e^x + C$

IBP: $\int u\,dv = uv - \int v\,du$

$u \xrightarrow{\text{diff}} du$

$dv \xrightarrow[\text{antidiff}]{} v$

Choices:

$u = x+1 \xrightarrow{\text{diff}} du = dx$

$dv = e^x\,dx \xrightarrow[\text{anti}]{} v = e^x$

$\int e^x\,dx$

$u = e^x \xrightarrow{\text{diff}} du = e^x\,dx$

$dv = (x+1)\,dx \xrightarrow[\text{antidiff}]{} v = \frac{1}{2}x^2 + x$

$\int \left(\frac{1}{2}x^2 + x\right)e^x\,dx$

ex $\int x^3 (\ln x)^2 \, dx$

IBP: $\int u\,dv = uv - \int v\,du$

$= \frac{1}{4}x^4 (\ln x)^2 - \int \frac{1}{4}x^4 \cdot 2\ln x \cdot \frac{1}{x}\,dx$

$= \frac{1}{4}x^4 (\ln x)^2 - \frac{1}{2}\int x^3 \ln x \, dx$

$\frac{x^4}{x} = x^3$

Choices:

$u = x^3 \xrightarrow{\text{diff}} du = 3x^2 dx$

$dv = (\ln x)^2 dx \xrightarrow[\text{antidiff}]{} v = ???$

$u = \ln x \rightarrow du = \frac{1}{x}dx$

$dv = x^3 dx \rightarrow v = \frac{1}{4}x^4$

$= \frac{1}{4}x^4 \ln^2 x - \frac{1}{2}\left[\frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \cdot \frac{1}{x}\,dx\right]$

$\begin{cases} u = (\ln x)^2 \xrightarrow{\text{diff}} 2\ln x \cdot \frac{1}{x}dx = du \\ dv = x^3 dx \xrightarrow[\text{antidiff}]{} \frac{1}{4}x^4 = v \end{cases}$

"dx" business: SubRule: NEED IT

IBP: eeehhh....

Shorthand: $dv = x^3 dx$

mean $\frac{dv}{dx} = x^3$

$$\frac{1}{4}x^4\ln^2 x - \frac{1}{2}\left[\frac{1}{4}x^4\ln x - \int \frac{1}{4}x^3\,dx\right]$$

$$= \frac{1}{4}x^4\ln^2 x - \frac{1}{2}\left(\frac{1}{4}x^4\ln x - \frac{1}{16}x^4\right) + C$$

$$= \frac{1}{4}x^4\ln^2 x - \frac{1}{8}x^4\ln x + \frac{1}{32}x^4 + C$$

Fancier IBP

ex $\int \ln x\,dx = ???$

$= \int \ln x \cdot 1\,dx$

$= x\ln x - \int x\cdot\frac{1}{x}\,dx$

$= x\ln x - \int 1\,dx$

$= \boxed{x\ln x - x + C}$

$u = \ln x \xrightarrow{\text{diff}} \frac{1}{x}\,dx = du$

$dv = 1\,dx \xrightarrow[\text{antid}]{} v = x$

Check:

$\frac{d}{dx}(x\ln x - x + C)$

$= x\cdot\frac{1}{x} + \ln x\cdot 1 - 1$

$= 1 + \ln x - 1 = \boxed{\ln x}$ ✓

ex $\int \arctan x \, dx$

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$$

$u = \arctan x \rightarrow du = \frac{1}{1+x^2} dx$

$dv = 1 \cdot dx \longrightarrow \quad v = x$

$$\int u\,dv = uv - \int v\,du$$

$= x \arctan x - \int \frac{x}{1+x^2} dx$

$= x \arctan x - \int \frac{1}{2} \cdot \frac{1}{w} \, dw$

$= x \arctan x - \frac{1}{2} \ln|w| + C$

$= \boxed{x \arctan x - \frac{1}{2} \ln|1+x^2| + C}$

substitution

$w = 1+x^2$

$\frac{dw}{dx} = 2x$, so $dw = 2x\,dx$

$\frac{1}{2} dw = x\,dx$

Definite $\int$: $\int_0^1 \arctan x \, dx$:

$$\left[x \arctan x - \tfrac{1}{2}\ln(1+x^2)\right]\Big|_0^1$$

$$\left[\arctan 1 - \tfrac{1}{2}\ln(2)\right] - \left[0 - \tfrac{1}{2}\cancel{\ln 1}^{0}\right]$$

$$= \boxed{\pi/4 - \tfrac{1}{2}\ln 2}$$

$$\int_0^1 \arctan x \, dx = x \arctan x \Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= 1 \cdot \arctan 1 - \cancel{0 \cdot \arctan 0} - \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \pi/4 - \int_0^1 \frac{x}{1+x^2}\,dx \;\ldots$$