Remember from Last Time:

The substitution rule helps us do the chain rule in reverse.

$$\frac{d}{dx}\left(\sin(x^3+\ln x)\right) = \cos(x^3+\ln x)\cdot\left(3x^2+\frac{1}{x}\right)$$

So:

$$\int \underbrace{\cos(x^3+\ln x)}_{u}\cdot\underbrace{\left(3x^2+\frac{1}{x}\right)dx}_{du} = \sin(x^3+\ln x)+C$$

$$\int \cos u\, du = \sin u + C = \sin(x^3+\ln x)+C$$

Similarly: $\int_1^2 \cos(x^3+\ln x)\left(3x^2+\frac{1}{x}\right)dx = \int_{1^3+\ln 1}^{2^3+\ln 2} \cos u\, du$

$$= \sin u\Big|_1^{8+\ln 2} = \sin(8+\ln 2)-\sin(1)$$

(ex) $\int_0^1 x e^{x^2} dx = \frac{1}{2}\int_0^1 e^u \, du = \frac{1}{2}\left[e^u \Big|_0^1 \right]$

$\boxed{u = x^2}$ *

If $x=0$, $u = x^2 = 0^2 = 0$

If $x=1$, $u = x^2 = 1$

$\frac{du}{dx} = 2x$

$\rightarrow du = 2x\,dx$

$\rightarrow \frac{1}{2} du = x\,dx$

$= \frac{1}{2}\left[e^1 - e^0\right] = \boxed{\frac{1}{2}(e-1)}$

(ex) $\int e^{x+e^x} dx = \int e^x e^{e^x} dx = \int e^u \, du = e^u + C$

$u = e^x$

$du = e^x dx$

$= \boxed{e^{e^x} + C}$

(ex) $\int x^5\sqrt{x^3+1}\,dx = \int x^3\cdot x^2\sqrt{x^3+1}\,dx$

$u = x^3+1 \longrightarrow x^3 = u-1$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$= \int x^3\sqrt{x^3+1}\,x^2\,dx$

$= \int (u-1)\cdot\sqrt{u}\cdot\frac{1}{3}\,du$

$= \frac{1}{3}\int \left(u^{3/2} - u^{1/2}\right) du = \frac{1}{3}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right] + C$

$= \frac{1}{3}\left[\frac{2}{5}(x^3+1)^{5/2} - \frac{2}{3}(x^3+1)^{3/2}\right] + C$

(ex) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$= \int \frac{-1}{u} \, du = -\ln|u| + C$

$= -\ln|\cos x| + C$

$= \ln|(\cos x)^{-1}| + C$

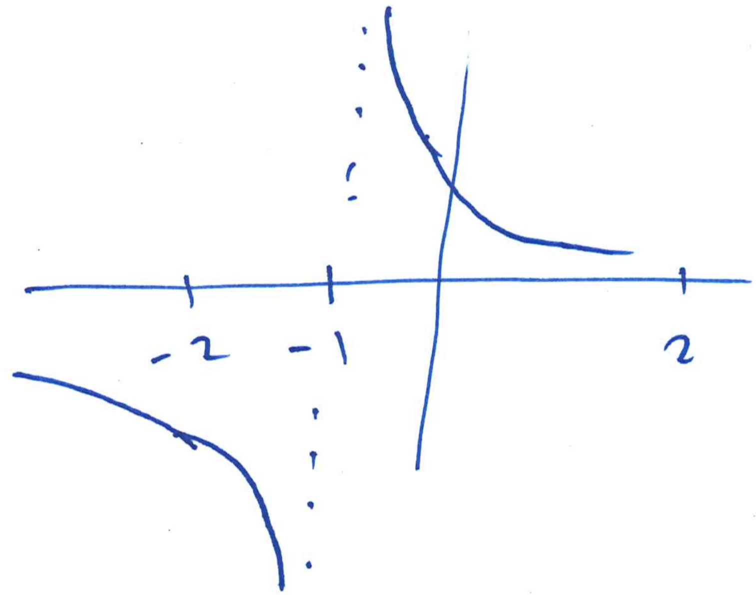
$= \ln\left|\frac{1}{\cos x}\right| + C$

$= \ln|\sec x| + C$

Recall:

$a \ln(b) = \ln(b^a)$

(ex) $\displaystyle\int_{-2}^{2} \frac{x^2}{x^3+1}\,dx$



Unclear here what "area under curve" even means

We'll talk about these later – for now, we have no tools to deal with this.

We can evaluate $\displaystyle\int \frac{x^2}{x^3+1}\,dx$ using $u = x^3+1$

We can also evaluate $\displaystyle\int_{0}^{2} \frac{x^2}{x^3+1}\,dx$

Ch 7.2 : Integration by Parts

Comes from product rule

$$\frac{d}{dx}\left(u(x) \cdot v(x)\right) = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

So : $$\int \left[u'(x)v(x) + u(x)v'(x)\right] dx = u(x)v(x) + C$$

$$\int u'(x)v(x)\,dx + \underline{\int u(x)v'(x)\,dx} = u(x)v(x) + C$$

$$\boxed{\int u(x)v'(x)\,dx = u(x)v(x) - \int v(x)u'(x)\,dx + C}$$

Antidiff prod of 2 fcns

Product Rule

u gets differentiated

v'(x) gets antidifferentiated

Mnemonic

$u = u(x)$

$\frac{du}{dx} = u'(x)$

$du = u'(x)dx$

$v' = v'(x)$

$v = \int v'(x)\,dx = v(x)$

$v = v(x)$

$v' = \frac{dv}{dx}$

$dv = v'(x)dx$

Integration by Parts:

$$\int u\,dv = uv - \int v\,du \quad + C$$

We antidiff a product of 2 fcns

call one u ⟶ differentiated

call other dv ⟶ antidifferentiated

(ex) $\int x \sin x \, dx$ No obvious substitution

$$u = x \xrightarrow{\text{diff}} du = dx$$

$$dv = \sin x \, dx \xrightarrow[\text{antidiff}]{} v = -\cos x$$

$$uv - \int v \, du = x(-\cos x) - \int (-\cos x) \, dx$$

$$= -x\cos x + \int \cos x \, dx$$

$$= \boxed{-x\cos x + \sin x + C}$$

Check: $\frac{d}{dx}\left(-x\cos x + \sin x + C\right) = (-x)(-\sin x) + \cos x(-1) + \cos x$

$$= x\sin x \quad \checkmark$$

(ex) $\int (x+1)\sec^2 x\, dx$

No obvious substitution

OPTION 1

$u = x+1 \xrightarrow{\text{diff}} du = dx$

$dv = \sec^2 x\, dx \xrightarrow{\text{antidiff}} v = \tan x$

OPTION 2

$u = \sec^2 x \xrightarrow{\text{diff}} 2\sec x \cdot \sec x \tan x$

$\frac{du}{dx} = 2\sec^2 x \tan x$

$dv = (x+1)dx \xrightarrow{\text{antidiff}} \frac{1}{2}x^2 + x = v$

$\int u\, dv = uv - \int v\, du$

$= uv - \int \tan x\, dx$

(*) NICER (*)

$= (x+1)\tan x - \int \tan x\, dx$

$= \boxed{(x+1)\tan x - \ln|\sec x| + C}$

$uv - \int v\, du$

$uv - \int (\frac{1}{2}x^2 + x) \cdot 2\sec^2 x \tan x\, dx$

(ex) $\int \underbrace{x}\,\underbrace{\ln x}\, dx$

$\int u\, dv = uv - \int v\, du$

$x, \ln x$:

choose one as $u \xrightarrow{\text{diff}} du$

other as $dv \xrightarrow[\text{antidiff}]{} v$

OPTION 1

$u = x$

$dv = \ln x\, dx$

$du = dx$

$v = \int \ln x\, dx = ?$

$u = \ln x$

$dv = x\, dx$

$du = \frac{1}{x} dx$

$v = \frac{1}{2}x^2$

$$\int x \ln x\, dx = \underbrace{\ln x}_{u} \cdot \underbrace{\tfrac{1}{2}x^2}_{v} - \int \underbrace{\tfrac{1}{2}x^2}_{v} \cdot \underbrace{\tfrac{1}{x}\, dx}_{du}$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\int x\, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2}\cdot\frac{1}{2}x^2 + C$$

$$= \boxed{\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C}$$

ex $\int x e^{6x}\,dx$

$$\int u\,dv = uv - \int v\,du$$

dv gets antidiff'd

u gets diff'd

$u = x \xrightarrow{\text{diff}} du = dx$

$dv = e^{6x}dx \xrightarrow[\text{antidiff}]{} v = \frac{1}{6}e^{6x}$

$u = e^{6x} \xrightarrow{\text{diff}} 6e^{6x}dx = du$

$dv = x\,dx \xrightarrow[\text{antidiff}]{} v = \frac{1}{2}x^2$

Option 1 → $\ldots - \int \frac{1}{6}e^{6x}dx$

$\ldots - \frac{1}{6}\int e^{6x}dx$ SIMPLER

Option 2 → $\ldots - \int \frac{1}{2}x^2 \cdot 6e^{6x}dx$

$\ldots - 3\int x^2 e^{6x}dx$ MORE COMPLICATED

$$uv - \int v\,du = x \cdot \frac{1}{6}e^{6x} - \int \frac{1}{6}e^{6x}dx$$

$$= \frac{1}{6}xe^{6x} - \frac{1}{6}\int e^{6x}dx$$

$$= \frac{1}{6}xe^{6x} - \frac{1}{36}e^{6x} + C$$

$$= \boxed{\frac{1}{6}e^{6x}\left(x - \frac{1}{6}\right) + C}$$

(ex) $\int (3t+5)\cos\left(\frac{t}{4}\right)dt$

no obvious sub → parts

$3t+5 \xrightarrow{\text{diff}}$ better (constant)

$3t+5 \xrightarrow{\text{antidiff}}$ slightly worse (quadratic)

$\cos\left(\frac{t}{4}\right) \xrightarrow{\text{diff}}$ / $\xrightarrow{\text{antidiff}}$ almost the same (diff constants)

} thought process for choosing u, dv in head

$u = 3t+5 \xrightarrow{\text{diff}} du = 3\,dt$

$dv = \cos\left(\frac{t}{4}\right)dt \xrightarrow{\text{antidiff}} v = 4\sin\left(\frac{t}{4}\right)$

$$\int u\,dv = uv - \int v\,du = (3t+5)\cdot 4\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right) 3\,dt$$

$$= 4(3t+5)\sin\left(\frac{t}{4}\right) - 12\int \sin\left(\frac{t}{4}\right)dt$$

$$= 4(3t+5)\sin\left(\frac{t}{4}\right) - 12\cdot 4\left[-\cos\left(\frac{t}{4}\right)\right] + C$$

$$= \boxed{4(3t+5)\sin\left(\frac{t}{4}\right) + 48\cos\left(\frac{t}{4}\right) + C}$$

Definite Integrals

$$\int_1^2 (3t+5)\cos\left(\frac{t}{4}\right)dt = 4(3t+5)\sin\left(\frac{t}{4}\right)\Big]_1^2 - 12\int_1^2 \sin\left(\frac{t}{4}\right)dt$$

$$= \left[4(6+5)\sin\left(\frac{2}{4}\right) - 4(3+5)\sin\left(\frac{1}{4}\right)\right] - 12\int_1^2 \sin\left(\frac{t}{4}\right)dt$$

$$= 44\sin\left(\frac{1}{2}\right) - 32\sin\left(\frac{1}{4}\right) - 12\left[-4\cos\left(\frac{t}{4}\right)\Big|_1^2\right]$$

$$= 44\sin\left(\frac{1}{2}\right) - 32\sin\left(\frac{1}{4}\right) + 48\left(\cos\left(\frac{1}{2}\right) - \cos\left(\frac{1}{4}\right)\right)$$

$$=$$