Remember from Last Time:

FTC (II):

If f is continuous on $[a,b]$ and F is any antiderivative of f on $[a,b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Vocab: $2x$ is the derivative of x^2

so x^2 is an antiderivative of $2x$

Written: $\int f(x)\,dx$ means: most general function whose derivative is $f(x)$

ex: $\int 2x\,dx = x^2 + C$

$\int f(x)\,dx$ "antideriv" "indefinite integral" } function

$\int_a^b f(x)\,dx$ "definite integral" } Area (number)

<u>Finding antiderivs</u>

Q: Given some fcn, what fcn has that as its derivative?

One way: "inspection" (stare at it)

ex $\int 2x\,dx = x^2 + C$

ex $\int \cos x\,dx = \sin x + C$

ex $\int \sin x\,dx = -\cos x + C$

ex $\int \frac{1}{x}\,dx = \ln|x| + C$

$f(x)$	$f'(x)$	$\int$ fact	more general $\int$ fact
x	1	$\int 1\,dx = x+C$	$\int a\,dx = ax+C$
x^2	$2x$	$\int 2x\,dx = x^2+C$	$\int x\,dx = \frac{1}{2}x^2+C$
x^3	$3x^2$	$\int 3x^2\,dx = x^3+C$	$\int x^2\,dx = \frac{1}{3}x^3+C$
x^4	$4x^3$		$\int x^3\,dx = \frac{1}{4}x^4+C$

Check:

$\frac{d}{dx}\left(\frac{1}{2}x^2+C\right)$

$= \frac{1}{2}(2x)+0 = x$ ✓

Check

$\frac{d}{dx}\left(\frac{1}{4}x^4+C\right)$

$= x^3$ ✓

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad \text{if } n \neq -1$$

$$\int x^{-1}\,dx = \ln|x| + C$$

Remember: $\frac{d}{dx}\left(a f(x) + b g(x) \right) =$

$a f'(x) + b g'(x)$

So: $\int (5x^2 - 15x + 3)\,dx$

$3 = 3x^0$

$= \int 5x^2\,dx - \int 15x\,dx + \int 3\,dx$

$= 5 \cdot \frac{x^3}{3} - 15\frac{x^2}{2} + 3x + C$

(ex) $\int 13\left(5x^{14} - 3x^{3/7} + 52e^{7x}\right)dx$

$= 13\left[5 \cdot \frac{x^{15}}{15} - 3\frac{x^{10/7}}{10/7} + 52\int e^{7x}\,dx\right]$

Q: What to do with $\int e^{7x}\,dx$

Careful: $e^{7x} \neq x^n$ so: $\int e^{7x}\,dx \neq \frac{e^{7x+1}}{7x+1}$

We can start by guessing:

$$\int e^x\,dx = e^x + C$$

$$\frac{d}{dx}\left(\frac{e^{7x}}{7}+C\right) = \frac{7e^{7x}}{7} \neq e^{7x}$$

$$\frac{d}{dx}\left(\underline{\frac{1}{7}e^{7x}+C}\right) = \frac{1}{7}(7)e^{7x} = \underline{e^{7x}}$$

So: $\boxed{13\left[\frac{x^{15}}{3} - \frac{21}{10}x^{10/7} + 52\left(\frac{1}{7}e^{7x}\right)\right] + C}$

$$\int 13 \cdot x^{14}\,dx$$

GUESS:

$$(x^{14})^{13}$$

$$\frac{d}{dx}\left[(x^{14})^{13}\right] = \frac{d}{dx}\left(x^{14\times 13}\right) = 14 \cdot 13\, x^{14\cdot 13 - 1}$$

Doesn't work ☹

GUESS: $\int \frac{1}{x}\,dx = \ln|x| + C$

MAYBE $\int \frac{1}{\sqrt{x}}\,dx \stackrel{?}{=} \ln\sqrt{x} + C$

Check: $\frac{d}{dx}\left(\ln\sqrt{x} + C\right) =$

$\frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x} \neq \frac{1}{\sqrt{x}}$

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ex: $f(x) = \sin(x^2+x)$

Then $f'(x) = \cos(x^2+x)\cdot(2x+1)$

So: $\int \cos(x^2+x)\cdot(2x+1)\,dx = \sin(x^2+x) + C$

Q: How do we recognize such an antiderivative?

Chain rule: "inside fcn" ; mult by deriv of "inside fcn"

Look for: $\nearrow$ $\nearrow$

(ex) $\int \underbrace{e^x}_{\text{deriv of inside fcn}} \sin(\underbrace{e^x}_{\text{inside fcn}})\,dx = -\cos(e^x) + C$

Check: $\frac{d}{dx}(-\cos(e^x)) = \sin(e^x)\cdot e^x$ ✓

This is good but hard - let's find a mnemonic

Chain Rule: $\frac{d}{dx}[f(u(x))] = f'(u(x))\cdot u'(x)$

So: $\int f'(u(x))\,u'(x)\,dx = f(u(x)) + C$

MNEMONIC: $u = u(x)$

$\frac{du}{dx} = u'(x)$ So: $du = u'(x)\,dx$ } change x's to u's w/o changing antiderivative ex

$$\int \underbrace{(3x^2+2x)}_{\substack{\text{deriv of}\\ x^3+x^2}} \sin \underbrace{(x^3+x^2)}_{\substack{\text{inside}\\ \text{fcn}}}\, dx$$

$u = x^3 + x^2$

$\frac{du}{dx} = 3x^2 + 2x$

$du = (3x^2+2x)\,dx$

use as dictionary to translate x's to u's (all of them, even dx)

$$= \int \sin\underbrace{(x^3+x^2)}_{u} \cdot \underbrace{(3x^2+2x)\,dx}_{du} = \int \sin u \, du = -\cos u + c$$

easier!

$$= \boxed{-\cos(x^3+x^2)+C}$$

Check:

$$\frac{d}{dx}\left(-\cos(x^3+x^2)+C\right) =$$

$$\sin\underbrace{(x^3+x^2)}_{\text{"u"}} \cdot \underbrace{(3x^2+2x)}_{\text{"du"}}$$

ex $\int \underbrace{\sin x}\, \underbrace{\cos x}\, dx$ du $= \int u \, du = \frac{1}{2}u^2 + C$

easier $= \boxed{\frac{1}{2}(\sin x)^2 + C}$

$\sin x = u$

$\frac{du}{dx} = \cos x$

$\hookrightarrow du = \cos x \, dx$

} dictionary

ex $\int e^{\sin x} \cos x \, dx$

One way: $u = \sin x$

$\frac{du}{dx} = \cos x$

$du = \cos x \, dx$

$\int e^u \, du$

$= e^u + C$

$= \boxed{e^{\sin x} + C}$

Another way: $u = e^{\sin x}$

$\frac{du}{dx} = e^{\sin x} \cdot \cos x$

$du = e^{\sin x} \cos x \, dx$

$\int du = \int 1 \, du$

$= u + C = \boxed{e^{\sin x} + C}$

ex $\int \frac{1 \cdot e^x}{e^x + 15} dx$ du $= \int \frac{1}{u} du$

$= \ln|u| + C = \boxed{\ln(e^x + 15) + C}$

$u = e^x + 15$

$\frac{du}{dx} = e^x$ so $du = e^x dx$

ex $\int x^4 (x^5 + 1)^8 \, dx = \int \frac{1}{5} \cdot u^8 \, du = \frac{1}{5} \cdot \frac{1}{9} u^9 + C$

$= \boxed{\frac{1}{45}(x^5 + 1)^9 + C}$

$u = x^5 + 1$

$du = 5x^4 dx$

$\frac{1}{5} du = x^4 dx$

Using substitution for <u>definite</u> integrals.

(ex) We found $\int x^4(x^5+1)^8\,dx = \int \frac{1}{5}u^8\,du = \frac{1}{45}(x^5+1)^9 + C$

So: $\int_0^1 x^4(x^5+1)^8\,dx \overset{FTC}{=} \frac{1}{45}(1^5+1)^9 - \frac{1}{45}(0^5+1)^9$

$= \boxed{\frac{1}{45}(2^9-1)}$ (area under curve)

Equivalent way: translate bounds as well as our fcns

$u = x^5+1$

$du = 5x^4\,dx$

$\frac{1}{5}du = x^4\,dx$

$x=0 \Rightarrow u = 0^5+1 = 1$

$x=1 \Rightarrow u = 1^5+1 = 2$

$$\int_{0=x}^{1=x} x^4(x^5+1)^8\,dx = \int_{1=u}^{2=u} u^8 \cdot \frac{1}{5}\,du = \frac{1}{45}u^9 \Big|_{u=1}^{u=2}$$

$\frac{1}{5}du$

$$= \frac{1}{45}\cdot 2^9 - \frac{1}{45}\cdot 1^9 = \boxed{\frac{1}{45}(2^9-1)}$$

(ex) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin^3 x}\,dx$

du

Tempting:

$u = \sin^3 x$

$du = 3\sin^2 x \cos x\,dx$

hard to make it work

Another try:

$u = \sin x$

$du = \cos x\,dx$

$x = \pi/4 \rightarrow u = \sin(\pi/4) = \frac{1}{\sqrt{2}}$

$x = \pi/2 \rightarrow u = \sin(\pi/2) = 1$

(ex)

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{u^3}\,du = \int_{1/\sqrt{2}}^{1} u^{-3}\,du = \left.\frac{u^{-2}}{-2}\right|_{1/\sqrt{2}}^{1}$$

$$= \frac{1^{-1}}{-2} - \frac{\left(\frac{1}{\sqrt{2}}\right)^{-2}}{-2}$$

$$= \frac{-1}{2} + \frac{1}{2}(2) = \frac{1}{2}$$

area under curve

ex $\displaystyle\int_4^8 \frac{s}{s-3}\,ds$

$u = s-3 \rightarrow s = u+3$

$\frac{du}{ds} = 1 \rightarrow du = ds$

$s=4$
$u = s-3$
$u = 4-3 = 1$

$s = 8$
$u = s-3 = 8-3 = 5$

$$\int_1^5 \frac{u+3}{u}\,du = \int_1^5 \frac{u}{u} + \frac{3}{u}\,du = \int_1^5 \left(1+\frac{3}{u}\right) du$$

$$= u + 3\ln|u| \Big|_1^5 = (5+3\ln 5) - (1+3\ln 1)$$

$$= 5+3\ln 5 - 1 - 0 = \boxed{4+3\ln 5}$$

ex $\displaystyle\int \frac{dx}{e^x+e^{-x}}$

Idea: $u = e^x + e^{-x}$

$du = (e^x - e^{-x})\,dx$

???

$$= \int \frac{1}{e^x+e^{-x}}\left(\frac{e^x}{e^x}\right)dx$$

$u = e^x \qquad du = e^x dx$

$$= \int \frac{\underbrace{e^x \, dx}_{du}}{(e^x)^2+1}$$

$$= \int \frac{1}{u^2+1}\,du = \arctan(u) + C = \boxed{\arctan(e^x) + C}$$