Remember from Last Time:

$\int f(x)\,dx$: "indefinite integral" (no bounds)

Meaning: the most general function whose derivative is $f(x)$. Also called "antiderivative"

For example: $\int 2x\,dx = x^2 + C$

because the derivative of $x^2 + C$ is $2x$.

Another example: $\int \sec^2 x\,dx = \tan x + C$

because the derivative of $\tan x + C$ is $\sec^2 x$.

From the Fundamental Theorem of Calculus Part 2, we have a relationship between definite and indefinite integrals: If $F(x)$ is any antiderivative of $f(x)$, then

$\int_a^b f(x)\,dx = F(b) - F(a)$ (as long as f is "nice")

More Antidifferentiation

Sometimes: solve by inspection (staring at it)

ex $\int \cos x \, dx = \sin x + C$

$\int \frac{1}{x} \, dx = \ln x + C$

Table

$f(x)$	$f'(x)$	Integral Facts	
x^2	$2x$	$\int 2x\,dx = x^2 + C$	$\int x\,dx = \frac{1}{2}x^2 + C$
x^3	$3x^2$	$\int 3x^2 dx = x^3 + C$	$\int x^2 dx = \frac{1}{3}x^3 + C$
x^4	$4x^3$	$\int 4x^3 dx = x^4 + C$	$\int x^3 dx = \frac{1}{4}x^4 + C$
			$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ if $n \neq -1$

ex) $\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = \frac{x^{1/2}}{1/2} + C = \boxed{2\sqrt{x} + C}$

Check:

$\frac{d}{dx}\left(2\sqrt{x} + c\right) =$

$2\left(\frac{1}{2\sqrt{x}}\right) + 0 = \frac{1}{\sqrt{x}}$

ex) $\int x^{1/3}\,dx = \frac{3}{4}x^{4/3} + C$

Caution:

$f(x) = \ln(\sqrt{x})$ <u>not</u> antider of $\frac{1}{\sqrt{x}}$

$f'(x) = \frac{1}{\sqrt{x}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2x} \neq \frac{1}{2\sqrt{x}}$

ex) $\int 5e^{6x}\,dx = \frac{5}{6}e^{6x} + C$

Start Guessing $\frac{d}{dx}\left(e^{6x}\right) = 6e^{6x}$

$\frac{d}{dx}\left(\underbrace{\frac{5}{6}}\cdot e^{6x}\right) = \frac{5}{6}\cdot 6e^{6x} = 5e^{6x}$

Differentiation :

$$\frac{d}{dx}\left(c\, f(x)\right) = c\,\frac{d}{dx}\left(f(x)\right)$$

$$\frac{d}{dx}\left(f(x)+g(x)\right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)$$

So: when we antidifferentiate,
- mult by const
- ~~add~~ adding functions

work "nicely" (no effects downstream)

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ex) $\int 13\left(5x^{14} - 3x^{3/7} + 52e^{7x}\right) dx$

$$= 13\left[5\cdot\frac{x^{15}}{15} - 3\frac{x^{10/7}}{10/7} + 52\cdot\frac{1}{7}e^{7x}\right] + C$$

$\frac{d}{dx}(e^{7x}) = 7e^{7x}$

So $\frac{d}{dx}\left(\frac{1}{7}e^{7x}\right) = e^{7x}$

So $\int e^{7x}dx = \frac{1}{7}e^{7x}$

Finding Antiderivs

For now: diff facts $\rightsquigarrow$ antidiff facts

eg $\frac{d}{dx}(x^2) = 2x$, so $\int 2x\,dx = x^2 + C$

ex $\frac{d}{dx}\left(\frac{1}{a}\arctan\left(\frac{x}{a}\right) + C\right) = \frac{1}{a}\cdot\frac{\frac{1}{a}}{1+\left(\frac{x}{a}\right)^2} = \frac{1}{a^2}\left(\frac{1}{1+\frac{x^2}{a^2}}\right)$

$= \frac{1}{a^2+x^2}$

So: $\boxed{\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C}$

ex $\frac{d}{dx}\left(\arcsin\left(\frac{x}{a}\right)+C\right)$

Recall: $\frac{d}{dx}\arcsin x$

$= \frac{1}{\sqrt{1-x^2}}$

$(a>0)$

$$= \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{1}{a}$$

$$= \frac{1}{\sqrt{1-\frac{x^2}{a^2}} \cdot \sqrt{a^2}} = \frac{1}{\sqrt{a^2-a^2\left(\frac{x^2}{a^2}\right)}}$$

$$= \frac{1}{\sqrt{a^2-x^2}}$$

So: $\boxed{\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right)+C}$

Substitution Rule ($\int$)
Comes from chain rule ($\frac{d}{dx}$)

eg: $\frac{d}{dx}\left(\sin(x^2+1)\right) = \cos(x^2+1) \cdot 2x$

So: $\int \cos(x^2+1) \cdot 2x \, dx = \sin(x^2+1) + C$

Chain rule gets us $f(\text{inside}) \cdot (\text{inside})'$

In generality: $\frac{d}{dx}\left[f(u(x)) \right] = f'(u(x)) \cdot u'(x)$

So: $\int f'(u(x)) u'(x) \, dx = f(u(x))$

Mnemonic to remember how it goes:

$u'(x) = \frac{du}{dx}$

So $u'(x) dx = \frac{du}{\cancel{dx}} \cancel{dx} = du$

Then: $\int f'(u(x))\underbrace{u'(x)\,dx}_{du} = \int f'(u)\,du = f(u)$

ex $\int \underbrace{(3x^2+2x)}\sin(\underbrace{x^3+x^2})\,dx$

$u = x^3+x^2$

$\frac{du}{dx} = 3x^2+2x$

So: $du = (3x^2+2x)\,dx$

translation dictionary

$x \rightarrow u$

$\int \underbrace{\sin(x^3+x^2)}_{\sin(u)} \cdot \underbrace{(3x^2+2x)\,dx}_{du} = \int \sin(u)\,du = -\cos u + C$

$= \boxed{-\cos(x^3+x^2)+C}$

ex $\displaystyle\int e^{\boxed{\sin x}} \underbrace{\cos x \, dx}_{du} = \int e^u \, du = e^u + C$

$= \boxed{e^{\sin x} + C}$

translation dictionary $x \to u$

$u = \sin x$

$\frac{du}{dx} = \cos x \longrightarrow du = \cos x \, dx$

ex $\displaystyle\int \underbrace{e^x \, dx}_{du} \sin(\underbrace{e^x}_{u}) = \int \sin(u) \, du = -\cos u + C = \boxed{-\cos(e^x) + C}$

$u = e^x$

$\frac{du}{dx} = e^x \longrightarrow du = \underline{e^x dx}$

ex $\displaystyle\int \underbrace{x^4 \, dx}_{\frac{1}{5}du} (\underbrace{x^5+1}_{})^8 = \int u^8 \cdot \frac{1}{5} \cdot du = \frac{u^9}{9 \cdot 5} + C = \boxed{\frac{1}{45}(x^5+1)^9 + C}$

$u = x^5 + 1$

$du = 5x^4 \, dx$

$\frac{1}{5} du = x^4 \, dx$

ex $\displaystyle\int \frac{4x+2}{x^2+x+1} \, dx = \int \frac{2}{u} \, du = 2\ln|u| + C$

$= \boxed{2\ln|x^2+x+1| + C}$

$u = x^2 + x + 1$

$\frac{du}{dx} = 2x + 1 \longrightarrow 2\left(\frac{du}{dx}\right) = 4x + 2$

$2du = (4x+2)\,dx$