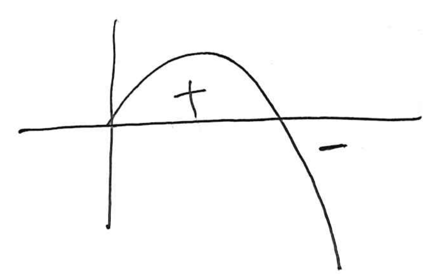
Remember from Last Time:

$$\int_a^b f(x)\,dx$$ is shorthand for

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x \quad , \quad \text{where } \Delta x = \frac{b-a}{n}$$

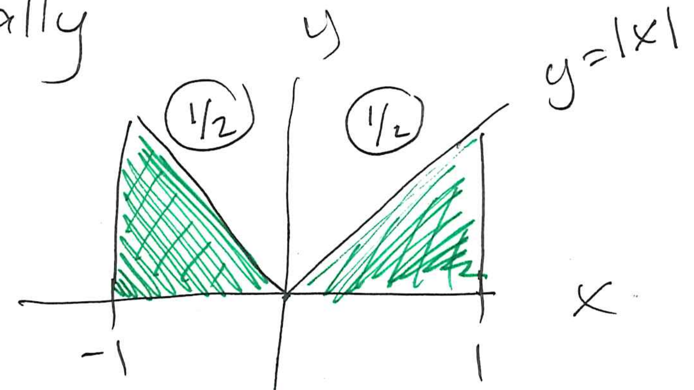
and this is the sum of rectangles' areas with base Δx and height $f(x)$

This gives the <u>net</u> area between $y=f(x)$ and the x-axis



Evaluating geometrically

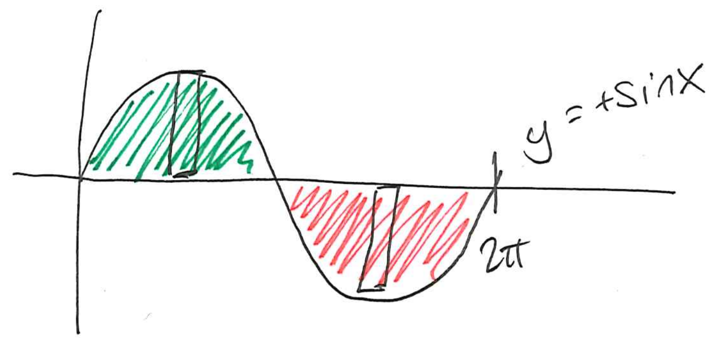
(ex) $\int_{-1}^{1} |x| \, dx = 1$



base: 1
height: 1

faster than Riemann Sums :)

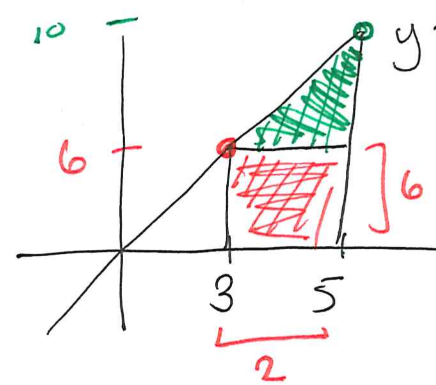
(ex) $\int_{0}^{2\pi} \sin x \, dx = 0$



Same area above & below x-axis

ex $\int_3^5 2x\,dx$

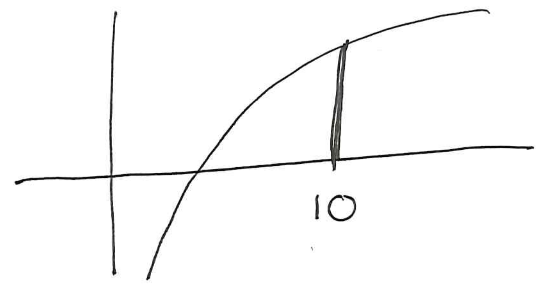
$= 16$



$y = 2x$] 4 $A: \frac{1}{2}(2)(4)$
$= 4$

] 6 $A: 12$

ex $\int_{10}^{10} \ln x \, dx = 0$



$\Delta x = \frac{10-10}{n} = 0$

width: 0

ex $\int_{-1}^{1} \sqrt{1-x^2}\,dx$

$= \frac{1}{2}\pi(1)^2$

$= \boxed{\pi/2}$

$y = \sqrt{1-x^2}$] (positive) $y \geq 0$

$\Rightarrow y^2 = 1 - x^2$

$x^2 + y^2 = 1$ unit circle

Review:

$x^2 = 4$: x could be 2 or -2

$\sqrt{4} = 2$ $\qquad$ $\pm\sqrt{4}$: ± 2

Find Area:

- limit of Riemann
  - easy concept (rectangles)
  - hard computation (limits of sums)
- geometry
- → Fundamental Thm of Calculus
  - tougher concept (why does it work??)
  - easier computations

$\boxed{\text{Ch } 5.3}$

Fundamental Theorem of Calculus (FTC), Part 1:

If f is continuous on $[a,b]$, then the area function

f: nice

$$A(x) = \int_a^x f(t)\,dt \quad , \quad a \le x \le b$$

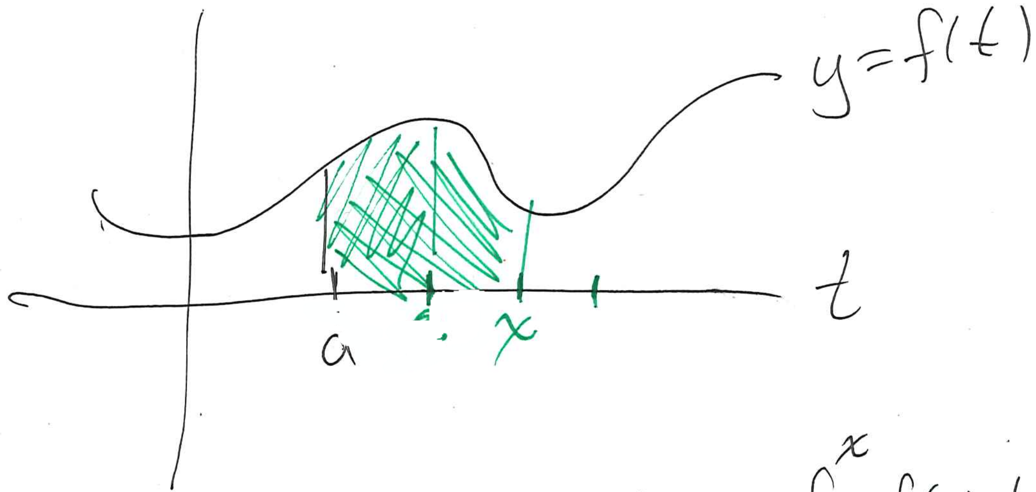
is continuous on $[a,b]$ and differentiable on (a,b). ← rarely used in Calc

Also: $A'(x) = f(x)$.

* Big Takeaway:
$\int$, deriv: kind of opposites

Notation:

We write $A(x) = \int_a^x f(t)\,dt$



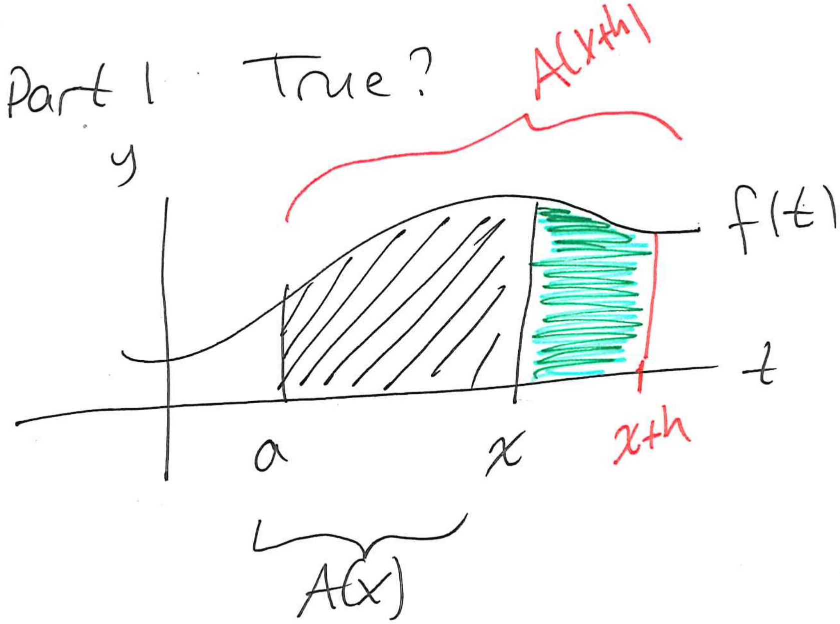
↑ like that

Tempting to write: $A(x) = \int_a^x f(x)\,dx$ ← not like this

But then: $A(3) = \int_a^3 f(3)\,d3$ ← ??? not a thing

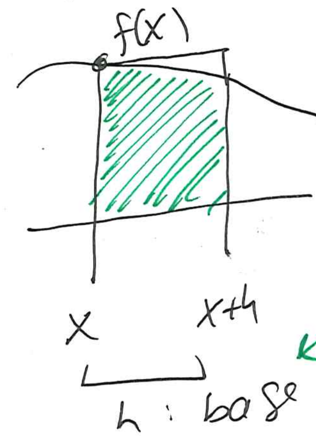
Why is FTC Part I True?

$A(x)$:



Green area: $A(x+h) - A(x)$

For small h:



Approx as rectangle, $h \cdot f(x)$ height

Definition of derivative:

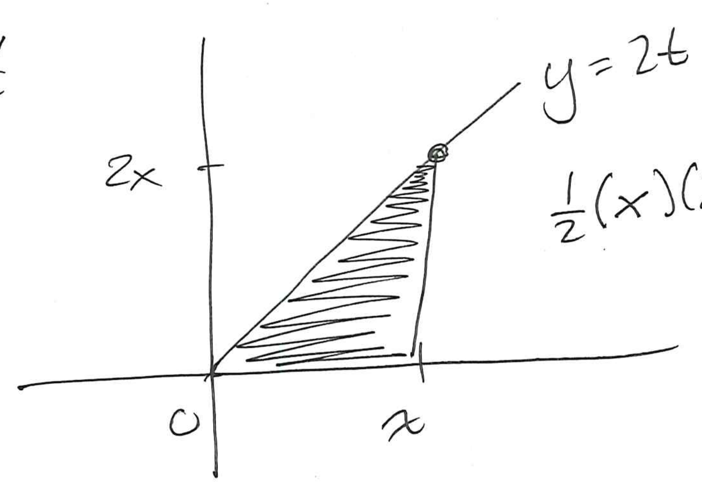
$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

$$= \lim_{h \to 0} \frac{h\, f(x)}{h}$$

$$= \lim_{h \to 0} f(x) = \boxed{f(x)}$$

That's why $A'(x) = f(x)$ FTC(I)

ex $$A(x) = \int_0^x 2t\,dt = \boxed{x^2}$$



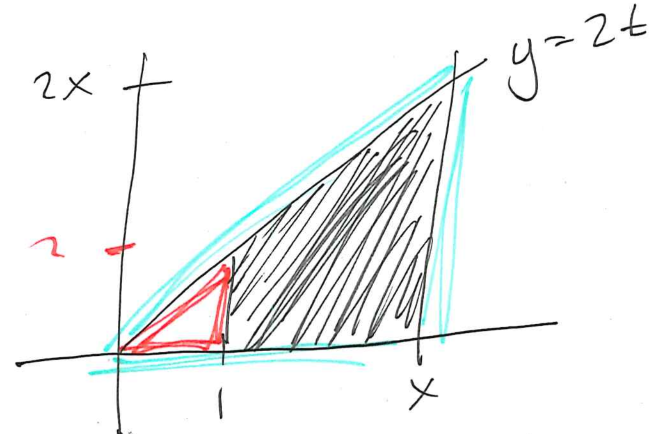
$\frac{1}{2}(x)(2x) = x^2$

Notice: $f(t) = 2t$

$A'(x) = 2x = f(x)$

ex $$B(x) = \int_1^x 2t\,dt = \boxed{x^2 - 1}$$

$B'(x) = 2x = f(x)$



Big ◁ − little ◁

$\frac{1}{2} \cdot 1 \cdot 2 = 1$

$\frac{1}{2}x \cdot 2x$

$x^2 - 1$

Many functions have same derivative.

FTC (I): $$A(x) = \int_a^x f(t)\,dt$$

is some function with derivative $f(x)$

Many different functions have same deriv, $f(x)$

Good news: functions w/ same derivative differ by a constant

eg

$x^2 \longrightarrow 2x$

$x^2-1 \longrightarrow 2x$

$x^2+1 \longrightarrow 2x$

$x^2+2 \longrightarrow 2x$

$x^2-\pi \longrightarrow 2x$

Suppose we want to find $A(x)$,
where $A(x) = \int_a^x f(t)\,dt$

FTC(I): Guess a function $F(x)$
that has $f(x)$ as its derivative.

Then $A'(x) = f(x)$ (FTC)
$F'(x) = f(x)$ (how I chose F)

So: $F(x) = A(x) + C$ for some constant C

Observe: $F(b) - F(a) = [A(b) + \cancel{C}] - [A(a) + \cancel{C}]$
$= A(b) - A(a)$
$$= \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{\Delta x = \frac{a-a}{n} = 0 \text{ no area}} = \int_a^b f(t)\,dt$$

FTC part 2:

If f is continuous on $[a,b]$ and F is any antiderivative of f on (a,b) $F' = f$

then $$\int_a^b f(x)\,dx = F(b) - F(a)$$

So: nice way to find area under curves!

ex) $$\int_3^{17} 2x\,dx = (17)^2 - (3)^2 = \underline{\underline{\text{Area}}}$$

(no Riemann required)

Need $F(x)$ whose deriv is $2x$

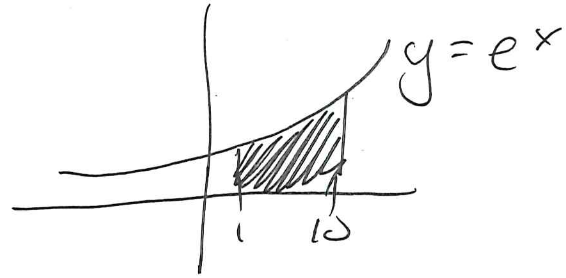
say, $F(x) = x^2$

ex

$$\int_1^{10} e^x \, dx$$

$$= F(10) - F(1)$$

$$= \boxed{e^{10} - e}$$



Need $F(x)$ with

$F'(x) = e^x$

use $F(x) = e^x$

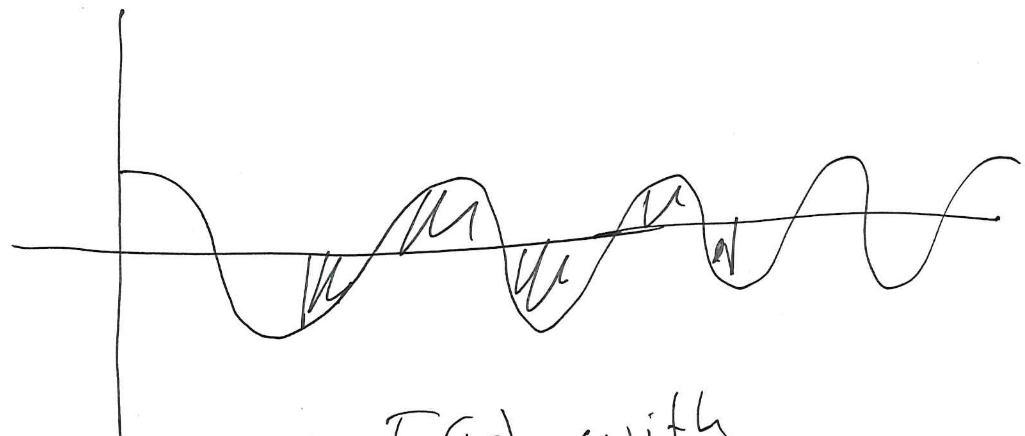
ex

$$\int_5^{10\pi} \cos x \, dx$$

$$= F(10\pi) - F(5)$$

$$= \sin(10\pi) - \sin(5)$$

$$= -\sin 5$$



Need $F(x)$ with

$F'(x) = \cos x$

$F(x) = \sin x$

The antiderivative of $f(x)$ is a function $F(x)$ with $f(x)$ as its derivative

That is: $F'(x) = f(x)$

we write: $\int f(x)\,dx$ "indefinite integral"
"antiderivative"

Fine print: lots of antiderivatives
we add "+C" to remind we can add any constant

ex: $\int 2x\,dx = x^2 + C$

ex: $\int \cos x\,dx = \sin x + C$

$\int f(x)\,dx$: function

$\int_a^b f(x)\,dx$: number (net area)

ex

Know: $\int \cos x \, dx = \sin x + C$

So: $\int 15 \cos x \, dx = 15 \sin x + C$

Also: $\int \cos(15x) \, dx = \frac{1}{15} \sin(15x) + C$

Guess: $\sin(15x)$

Check: $\frac{d}{dx}\left(\sin(15x)\right)$

$= 15 \cos(15x)$

Guess: $\frac{1}{15} \sin(15x)$

Deriv: $\frac{\cancel{1}}{\cancel{15}}\left[\cancel{15} \cos(15x)\right]$

$\int \cos x \, dx$: function whose derivative is $\cos x$

$\int 2x \, dx$: function whose derivative is $2x$