Riemann Sums:

$$\sum_{k=1}^{n} \Delta x \, f(x_k^*) = \underset{\Delta x}{\Box}\, f(x_1^*) + \underset{\Delta x}{\Box}\, f(x_2^*) + \cdots + \underset{\Delta x}{\Box}\, f(x_n^*)$$

$(k=1)$ $\qquad$ $(k=2)$ $\qquad$ $(k=n)$

Right RS: $x_k^* = a + k\Delta x$ $\qquad$ $\Delta x = \frac{b-a}{n}$

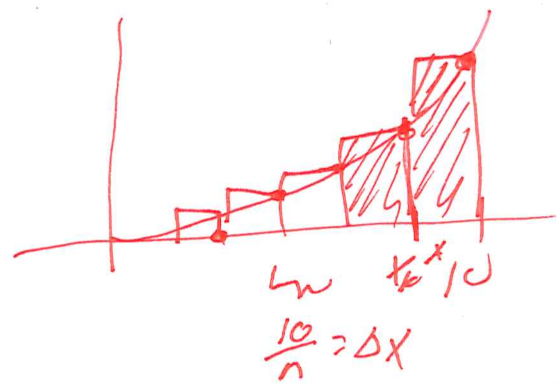
Left RS: $x_k^* = a + (k-1)\Delta x$

Midpt RS: $x_k^* = a + (k-\frac{1}{2})\Delta x$

(ex) Right RS $\quad f(x) = \frac{1}{2}x^3$, $\quad [0, 10]$, $\quad n$ rectangles

$\Delta x = \frac{b-a}{n} = \frac{10-0}{n} = \frac{10}{n}$

$x_k^* = a + k\Delta x = 0 + k(\frac{10}{n}) = \frac{10}{n}k$



$$\sum_{k=1}^{n} \underbrace{\frac{10}{n}}_{\Delta x} f(\underbrace{\frac{10}{n}\cdot k}_{x_k^*}) = \sum_{k=1}^{n} \frac{10}{n} \cdot \frac{1}{2}\left[\frac{10}{n}k\right]^3$$

$$= \sum_{k=1}^{n} \underbrace{\frac{1}{2}\cdot\left(\frac{10}{n}\right)^4}_{\text{factor out}} \cdot k^3 = \frac{1}{2}\left(\frac{10}{n}\right)^4 \sum_{k=1}^{n} k^3$$

p340 formula

$$= \frac{1}{2}\left(\frac{10}{n}\right)^4 \cdot \underbrace{\frac{n^2(n+1)^2}{4}}_{\text{(formula)}} = \frac{10^4 (n+1)^2}{8\, n^2}$$

eg RS with 10 rectangles: $\frac{10^4 \cdot 11^2}{8 \cdot 10^2}$

RS with 100 rectangles: $\frac{10^4 (101)^2}{8 \cdot 100^2}$

$$\underbrace{\frac{1}{2}\left(\frac{10}{n}\right)^4}_{(k=1)} \cdot 1^3 + \underbrace{\frac{1}{2}\left(\frac{10}{n}\right)^4}_{(k=2)} \cdot 2^3 + \underbrace{\frac{1}{2}\left(\frac{10}{n}\right)^4}_{(k=3)} \cdot 3^3 + \cdots$$

$$= \frac{1}{2}\left(\frac{10}{n}\right)^4 \left[1^3 + 2^3 + 3^3 + \cdots \right] = \frac{1}{2}\left(\frac{10}{n}\right)^4 \cdot \sum_{k=1}^{n} k^3$$

RS with ∞ rectangles:

n rectangles RS: $\dfrac{10^4(n+1)^2}{8n^2}$

$$\lim_{n\to\infty}\left[\frac{10^4(n+1)^2}{8n^2}\right] = \frac{10^4}{8}\cdot\frac{1}{1} = \frac{10^4}{8}$$

Rational fcn $\left(\frac{\text{polynom}}{\text{polynom}}\right)$

Deg of num = deg of denom

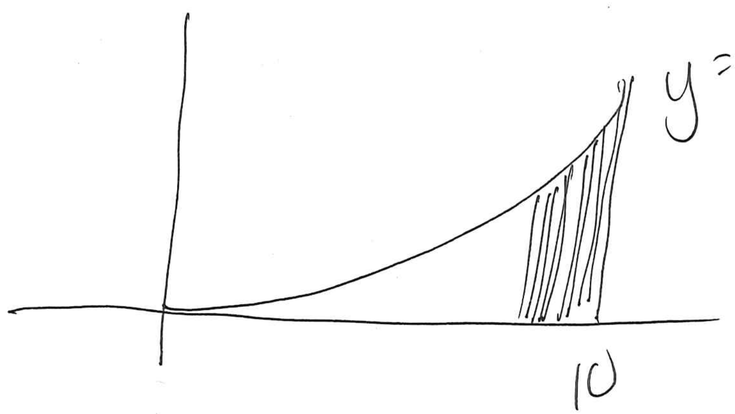
Limit $(n\to\infty)$: ratio of leading coefficients

$$\frac{(n+1)^2}{n^2} = \frac{n^2+2n+1}{n^2}$$

deg: 2

deg: 2

$\frac{10^4}{8}$: exact area under $f(x)=\frac{1}{2}x^3$, $[0,10]$



ex Use lim of Riemann Sum to find exact area under $f(x)=x^2$ over $[0,1]$.
a b

Can use any RS; same answer (Right RS usually easiest)

Right RS: $\sum_{k=1}^{n} \Delta x\, f(a+k\Delta x)$

where $\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

(plugging in)

$$\sum_{k=1}^{n} \underbrace{\frac{1}{n}}_{\Delta x} \cdot f\left(0+k\cdot\frac{1}{n}\right) = \sum_{k=1}^{n} \frac{1}{n}\left(k\cdot\frac{1}{n}\right)^2$$

$\frac{1}{n} \cdot k^2 \cdot \frac{1}{n^2}$

(simplify, evaluate)

$$= \sum_{k=1}^{n} \left(\underbrace{\frac{1}{n^3}}_{\text{factor}} \cdot k^2\right) = \frac{1}{n^3} \sum_{k=1}^{n} k^2$$

formulas p340

$$= \frac{1}{n^3}\, \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6n^2}$$

Using n rectangles ← approx of area under curve

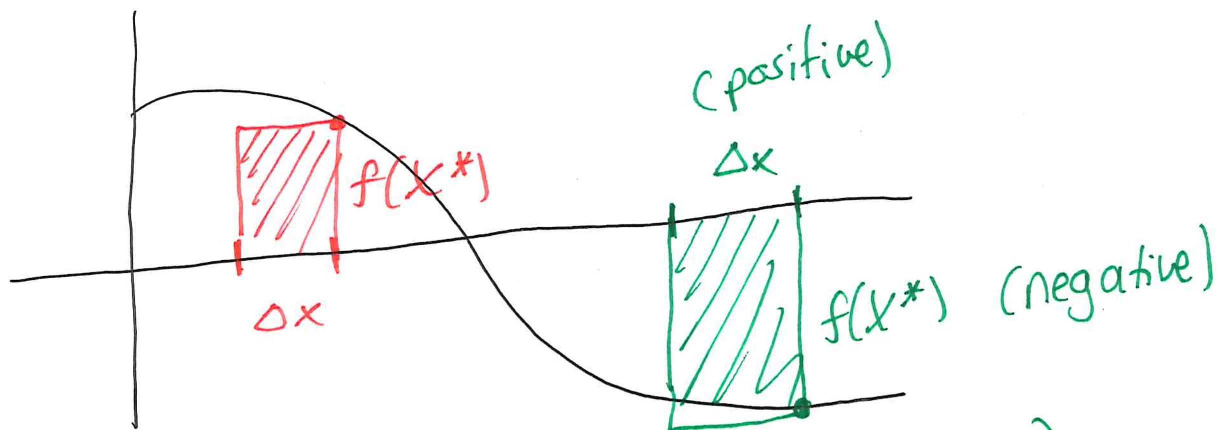
∞ rectangles:

$$\lim_{n\to\infty}\left(\frac{2n^2+3n+1}{6n^2}\right) = \frac{2}{6} = \boxed{\frac{1}{3}}$$

rectangles

Exact area under $y=x^2$, $x=0$ to $x=1$

Q: What if $f(x) < 0$?



$-$ (Area btw curve and x-axis)

Actually compute "net area"

(Area above axis)
$-$ (Area below axis)

Shorthand Notation

(net) Area btw $y=f(x)$ and axis, $a \le x \le b$:

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Delta x f(x_k^*)$$, where $\Delta x = \frac{b-a}{n}$

Shorthand:

$$\int_a^b f(x)\, dx$$

$\int$, Σ both "S" for "sum"

"definite integral"

Properties of a Definite Integral

$\lim \sum \Delta x \, f(x_k^*)$

$\Delta x = \frac{b-a}{n}$

① $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$\Delta x : \frac{b-a}{n}$

$\Delta x : \frac{a-b}{n} = -\left(\frac{b-a}{n}\right)$

Δx Δx

② $$\int_a^b c \cdot f(x)\,dx = c \int_a^b f(x)\,dx$$ (factor out constant)

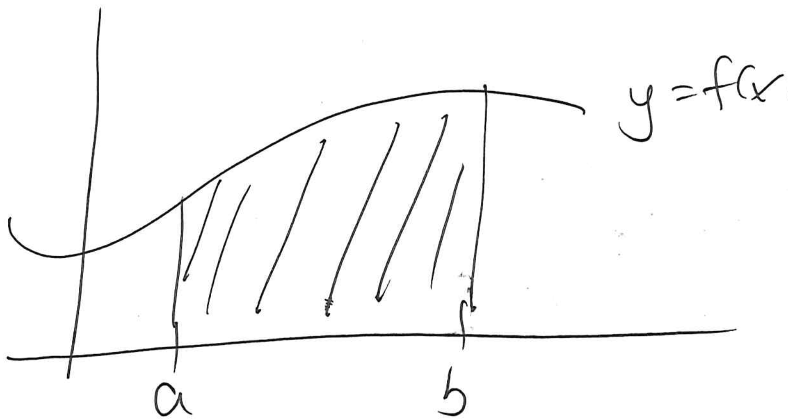
↑ constant

③ $$\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

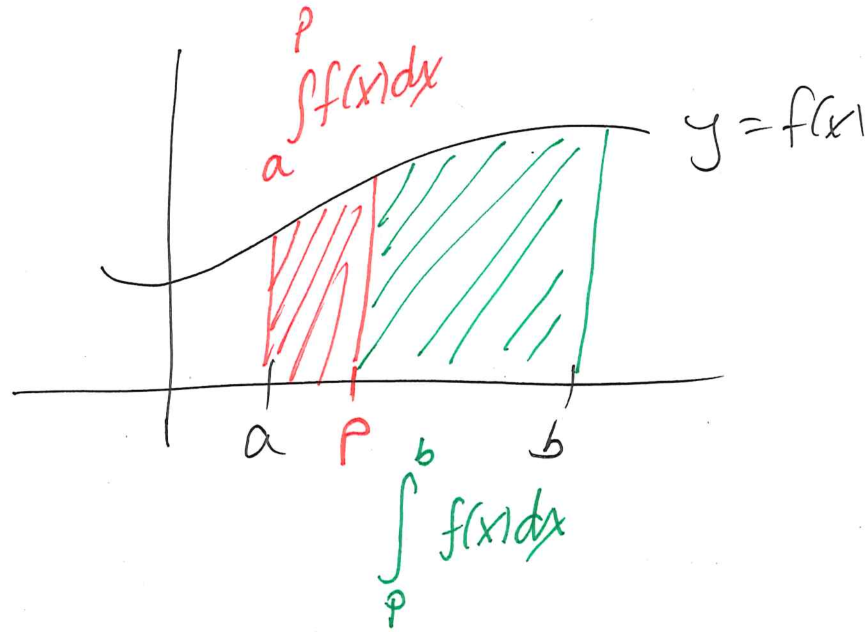
$$\sum (A+B) = (A+B) + (A+B) + (A+B) + \cdots (A+B)$$
$$= (A+A+A+\cdots+A) + (B+B+B+\cdots+B)$$
$$= \sum A + \sum B$$

(4) For any point p,

$$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$$



$\int_a^b f(x)\,dx$



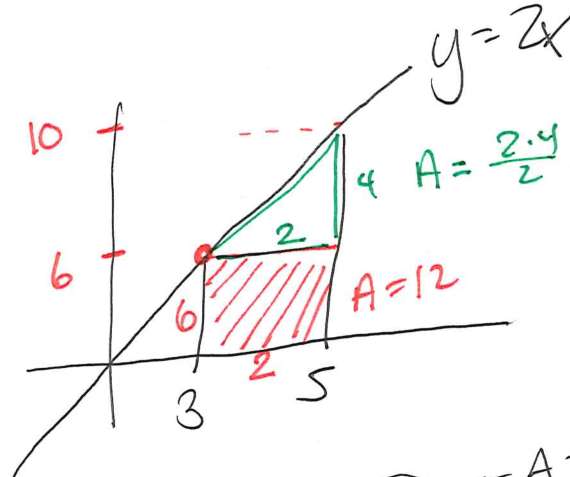
Area under Curve

∘ lim of Riemann Sums

Other ways as well.

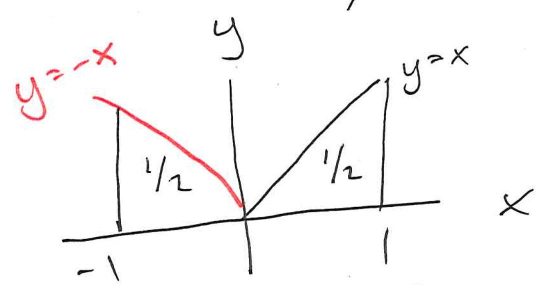
$\boxed{\text{Geometry}}$

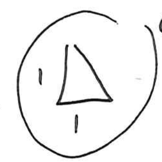
$$\int_3^5 2x\,dx = 12+4 = 16$$



$A = \frac{2\cdot 4}{2} = 4$

ex $\int_{-1}^{1} |x|\,dx = 1$



2 

← $A = \frac{1}{2}$

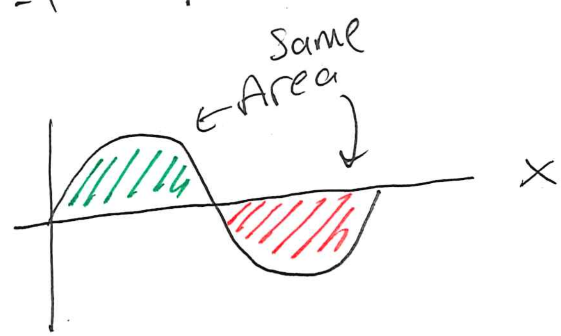
$\Delta x\, f(x)$

$\Delta x = \frac{1-(-1)}{n}$

(pos)

$f(x)$: pos

ex $\int_0^{2\pi} \sin x\,dx = 0$

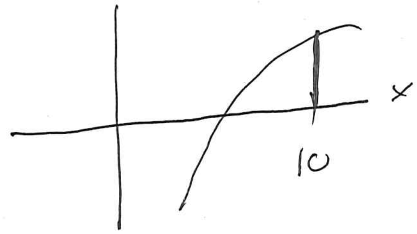


$A - A = 0$

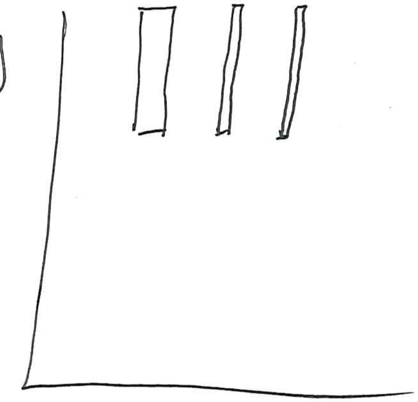
$\Delta x\, f(x)$

$f(x)$ sometimes negative

(ex) $\int_{10}^{10} \ln x \, dx$
$= 0$



← infinitely thin



(ex) $\int_{-1}^{1} \sqrt{1-x^2} \, dx$

$= \frac{1}{2}\pi(1)^2$

$= \pi/2$

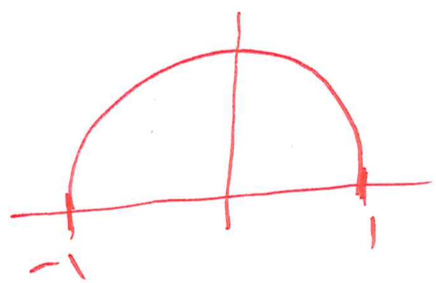
Note:

$y = \sqrt{1-x^2}$

$\Rightarrow y^2 = 1-x^2$

$\Rightarrow y^2 + x^2 = 1$ unit circle

$y = \pm\sqrt{1-x^2}$



unit circle upper half
$(y \geq 0)$

Area under Curve

- Riemann sum limit
  - Conceptually easy (cutting into ▯s)
  - Computationally rough!
- Geometry (rectangles, triangles, circles, symmetry)
  - nice when it's available
  - not always!
- Up Next: Fundamental Theorem of Calculus
  - Conceptually tricky
  - Computationally better

Ch 5.3

Fundamental Theorem of Calculus, Part I:

If f is continuous on $[a,b]$, then the area function } fine print

$$A(x) = \int_a^x f(t)\,dt \quad \text{for } a \le x \le b$$

is continuous on $[a,b]$ and differentiable on (a,b). } fine print

And: $A'(x) = f(x)$ ✱ big takeaway

Note about Notation:

Why not: $A(x) = \int_a^x f(x)\,dx$ y

$A(3)$ should be area $[a, 3]$, $\int_a^3 f(x)\,dx$

But using bad way

$$A(3) = \int_a^3 f(3)\,d3 \quad ???$$

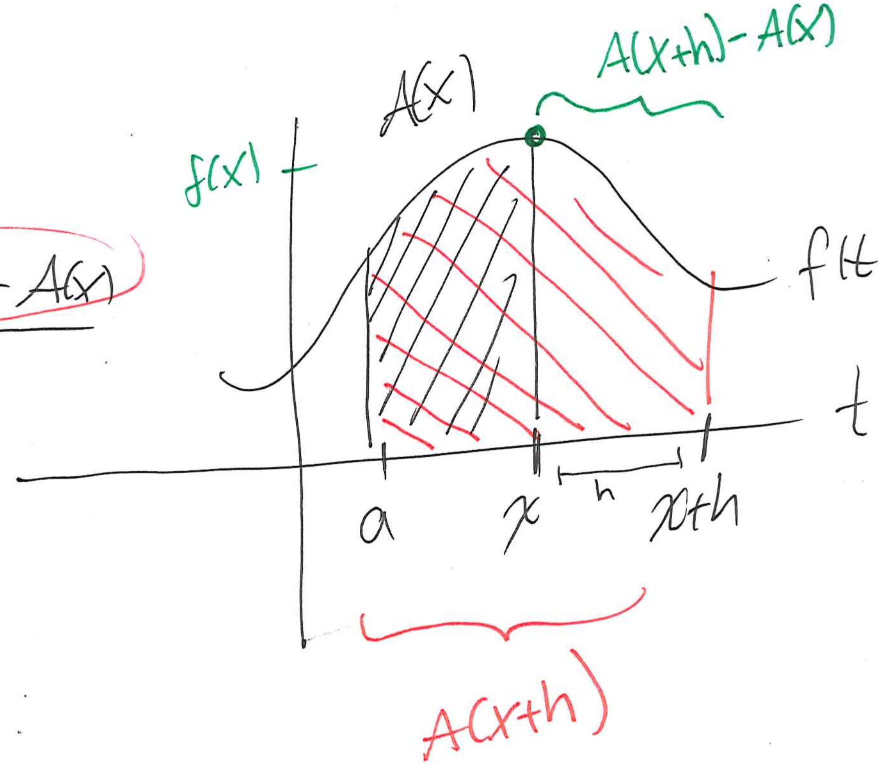
What it tells us: There is a close relationship between integrals & derivatives.

(kind of opposites: (diff int) gets original function back)

More in Part 2

Why is FTC(I) True?

$$A(x) = \int_a^x f(t)\,dt$$



$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

def of deriv

$$= \lim_{h \to 0} \frac{h f(x)}{h}$$

Area $\underbrace{\text{base}}_{h} \cdot \underbrace{\text{height}}_{f}$

$$= \lim_{h \to 0} f(x) = f(x)$$

FTC: $A'(x) = f(x)$

For h small, approx area by rectangle (like in Riemann sums!)

Base: h

Height $f(x)$ (left endpt)

Observation:

$$A(x) = \int_a^x f(t)\,dt, \quad \text{then}$$

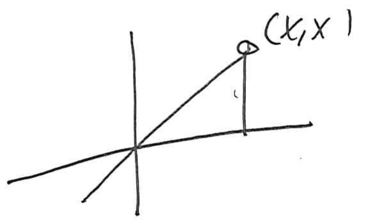
$A(x)$ is a function whose derivative is $f(x)$ (FTC I)

Lots of functions have same derivative.

ex

$$\int_0^x t\,dt = \frac{1}{2}x \cdot x = \boxed{\frac{1}{2}x^2}$$

Note: $\frac{d}{dx}\left(\frac{1}{2}x^2\right) = x$



ex

$$\int_1^x t\,dt = \boxed{\frac{1}{2}x^2 - \frac{1}{2}}$$

Note:

$$\frac{d}{dx}\left(\frac{1}{2}x^2 - \frac{1}{2}\right) = x$$

If two functions have same derivative, then they only differ by a constant

e.g. $f(x) = \frac{1}{2}x^2$

$g(x) = \frac{1}{2}x^2 + \frac{1}{2}$

Suppose we want to find $A(x)$,

where $A(x) = \int_a^x f(t)\,dt$.

FTC (I) $\Rightarrow$ we want a function whose derivative is $f(x)$.

Choose <u>any</u> function $F(x)$ whose deriv is $f(x)$

(maybe $F \neq A$)

But: there's some constant C such that

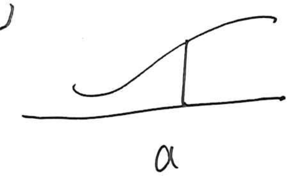
$F(x) = A(x) + C$

Observation:

$$F(b)-F(a) = [A(b)+\cancel{C}] - [A(a)+\cancel{C}]$$

$$= A(b) - A(a)$$

$$= \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{\text{no area: } = 0}$$



$$= \int_a^b f(t)\,dt$$

$\Rightarrow$ FTC (II)

Fundamental Theorem of Calculus, Part II:

If f is continuous on $[a,b]$ and

F is any antiderivative of f on $[a,b]$

then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

$\Rightarrow$ Computationally easy way to find area under curves

(concept more difficult than Riemann Sums)

(ex) $\displaystyle\int_3^5 x^2\,dx$

Notice: $\frac{1}{3}x^3 = F(x)$

$F'(x) = x^2 = f(x)$

$= \frac{1}{3}(5)^3 - \frac{1}{3}(3)^3$

$= \boxed{\dfrac{5^3 - 3^3}{3}}$ exact area