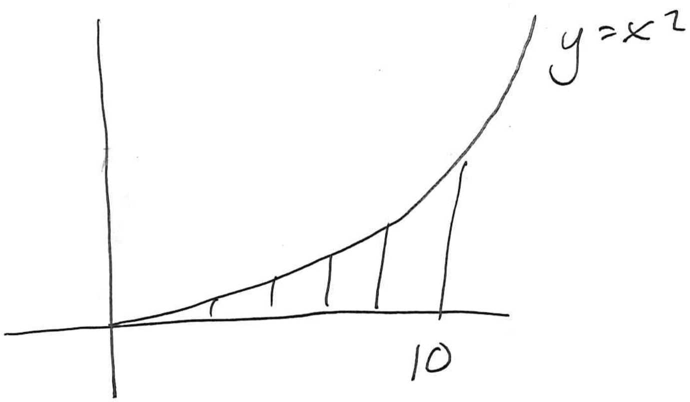
ex Use a Riemann sum to find the exact area under the curve $y = x^2$ on the interval $[0, 10]$

(Last time I made a copy error at the end ☹)



General Form:

(weren't told which RS, right RS easiest!)

$$\sum_{k=1}^{n} f(a + k\Delta x)\cdot \Delta x$$

$\Delta x = \frac{b-a}{n}$ base

$f(x)$

Δx

$\Delta x = \frac{10-0}{n} = \frac{10}{n}$

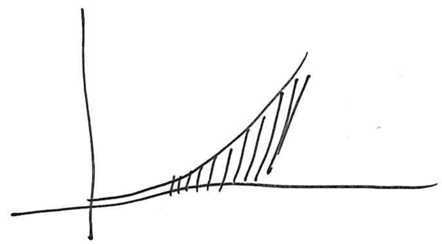
$$\sum_{k=1}^{n} \left(0 + k\cdot\frac{10}{n}\right)^2 \cdot \underbrace{\frac{10}{n}}_{\Delta x}$$

$$= \sum_{k=1}^{n} k^2 \left(\frac{10}{n}\right)^2 \frac{10}{n} \underset{\text{factor}}{=} \frac{10^3}{n^3}\left(\sum_{k=1}^{n} k^2\right)$$

p340 formula

$$\frac{10^3}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) = \frac{10^3}{6}\cdot\frac{(n+1)(2n+1)}{n^2}$$

Imagine using ∞ rectangles



$$\lim_{n\to\infty}\underbrace{\left[\frac{10^3}{6}\,\frac{(n+1)(2n+1)}{n^2}\right]}_{\text{Right RS}}$$

$$= \lim_{n\to\infty} \frac{10^3}{6}\cdot\left(\frac{2n^2+3n+1}{n^2}\right) = \frac{10^3}{6}\cdot 2 = \boxed{\frac{10^3}{3}}$$

exact area under curve

Imagine using ∞ rectangles

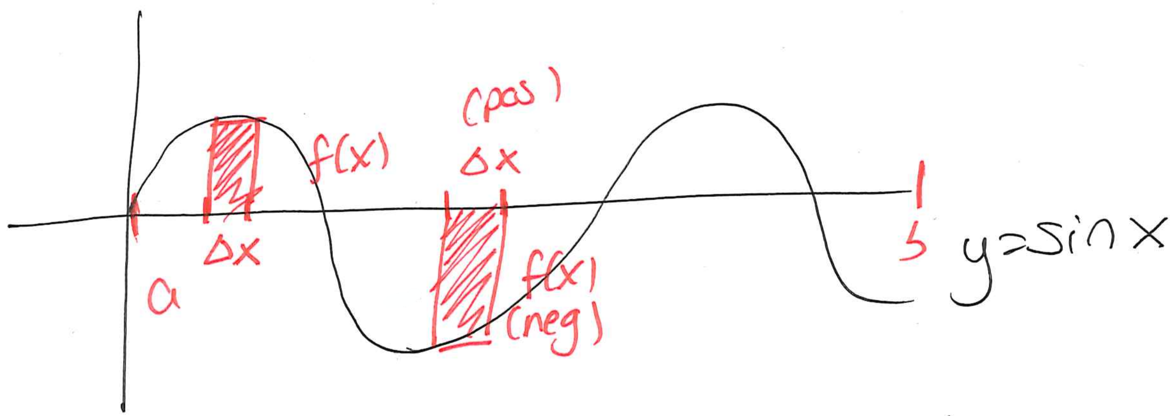
$$\underbrace{\lim_{n\to\infty} \frac{1}{8} \cdot \left(\frac{n+1}{n}\right)^2}_{RS} = \frac{1}{8}\left(\frac{1}{1}\right) = \boxed{\frac{1}{8}}$$

exactly $\frac{1}{8}$

What if f(x) is negative?

$\Delta x = \frac{b-a}{n}$ base

f(x): height



We're actually finding "net area"

[Area above x-axis] − [Area below x-axis] (ex) Use the limit of RS to find the exact area under the curve $y = \frac{1}{2}x^3$, over $[0, 1]$

General form:

$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$

$a = 0$

$$\sum_{k=1}^{n} f(a + k\Delta x) \cdot \Delta x =$$

$$\sum_{k=1}^{n} \underbrace{f(0 + k(\tfrac{1}{n}))}_{\text{height}} \cdot \underbrace{\tfrac{1}{n}}_{\text{base}} = \sum_{k=1}^{n} \frac{1}{2}\left(k\left(\frac{1}{n}\right)\right)^3 \cdot \frac{1}{n}$$

$$= \sum_{k=1}^{n} \frac{1}{2n^4} \cdot k^3 \underset{\text{factor}}{=} \frac{1}{2n^4} \sum_{k=1}^{n} k^3 = \frac{1}{2n^4} \cdot \frac{n^2(n+1)^2}{4}$$

formula

$$= \frac{1}{8} \cdot \frac{(n+1)^2}{n^2}$$

We're computing:

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

$\Delta x = \frac{b-a}{n}$

Shorthand notation:

$$\int_a^b f(x)\,dx$$

bounds

Σ, $\int$ "s" for "sum"

"definite integral"

$\Delta x = \frac{b-a}{n}$

Properties:

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

vs

$\Delta x = \frac{a-b}{n} = -\frac{b-a}{n}$

2. If c constant, $\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$

factor

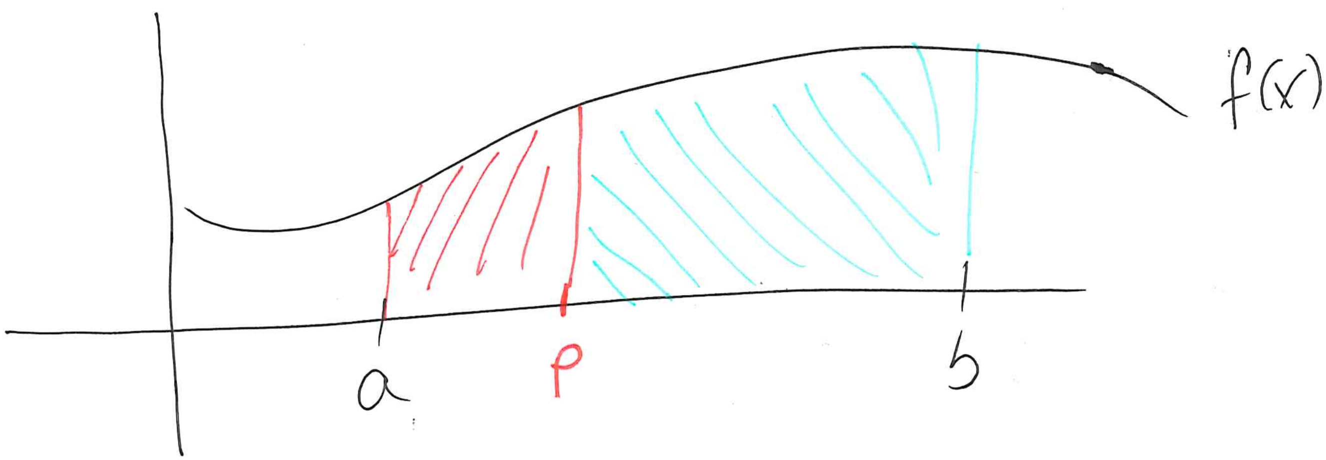
③ $$\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

From algebra w/ Σ:

$$\sum_{k=1}^{4} [f(k) + g(k)] = f(1) + g(1) + f(2) + g(2) + f(3) + g(3) + f(4) + g(4)$$

$$\sum_{k=1}^{4} f(k) + \sum_{k=1}^{4} g(k)$$

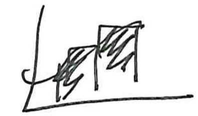
④ $$\int_a^b f(x)\,dx = \int_a^p f(x)\,dx + \int_p^b f(x)\,dx$$



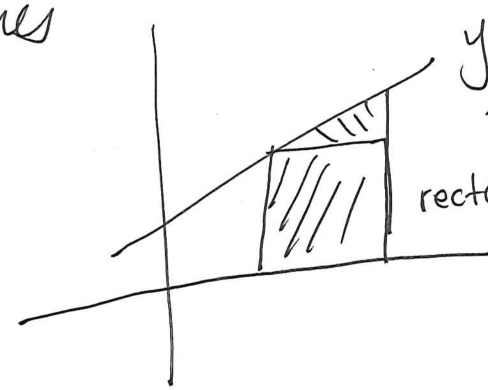
How can we evaluate?

- Riemann sum + limit

  conceptually easy
  computationally hard



- Geometry works sometimes



- Next time:
  Fundamental Theorem
  conceptually difficult
  computationally easy